ON THE TOPOLOGY OF CYCLIC PRODUCTS OF SPHERES

BY

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INTRODUCTION

We propose to study certain topological properties of cyclic products of spheres. Applications may be made in the theory of fiber bundles(1). Our success is as follows.

We shall introduce first the notion of "regular imbedding of a space into its Γ-products" (this notion appears in a more general situation). Let X be a locally compact, paracompact Hausdorff space. By the Γ-product X^Γ of X we mean the orbit space over a q-fold Cartesian product X^q of X on which a group Γ of permutations of factors of X^q acts. It is the q-fold cyclic product or q-fold symmetric product if Γ is a cyclic group of order q or the whole symmetric group. Let $s_0 \in X$ be any given point. By identifying any $x \in X$ with $\bar{I}(x, s_0, s_0, \cdots, s_0) \in X^\Gamma$ where $\bar{I}$ denotes the natural map: $X^q \to X^\Gamma$, X is topologically imbedded in X^Γ. We call this imbedding a regular imbedding. Let G be a coefficient group for cohomology groups. We show then that every cohomology class $v \in H^s(X, G)$ extends to a certain cohomology class $\mu(v) \in H^s(X^\Gamma, G)$ such that μ: $H^s(X, G) \to H^s(X^\Gamma, G)$ is an into-isomorphism and $\eta\mu$ is the identity isomorphism of $H^s(X, G)$ where η: $H^s(X^\Gamma, G) \to H^s(X, G)$ is the projection homormorphism.

In general, no more cohomology relations between X and X^Γ are studied in this paper. We consider the p-fold cyclic product, to be denoted by ϑ_{np} of an n-sphere S_n with p prime and $n \geqq 2$. ϑ_{np} has vanishing integral cohomology groups $H^s(\vartheta_{np}, Z)$ of dimensions between 0 and n and has infinite cyclic n-dimensional integral cohomology group $H^n(\vartheta_{np}, Z)$. If S_n is regularly imbedded in ϑ_{np}, the into-isomorphism μ above maps a generator of $H^n(S_n, Z)$ into a generator g_n^* of $H^n(\vartheta_{np}, Z)$. We shall calculate in ϑ_{np} explicitly the iterated cyclic reduced powers mod p of g_n^*. In particular, we have that $\mathcal{P}^k g_n^* \neq 0$ for all $2k \leqq n$.

In the last chapter of this paper, we study certain homotopy properties of 2-fold and 3-fold cyclic products of spheres. These are partial results on the problem of "symmetrization killing homotopy groups" as we state in

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(1) For instance, in the author's paper, *On the theory of obstructions of fiber bundles* (Thesis, University of Chicago, 1952; to appear in the Ann. of Math.), a fiber bundle $\mathcal{B}^\#$ related to a given sphere bundle $\mathcal{B}$ is constructed with fiber the 2-fold cyclic product of a sphere to deduce the formula of secondary obstructions of $\mathcal{B}$. The notion "regular imbedding" in §3 and the results $\pi_i(S_n * S_n) = 0$ etc. for $i = n+1$, $n+2$ in (13.3), (13.6) of §13 are drawn from the author's thesis.

§10. We show that $\pi_i(\vartheta_{n3})$ vanishes for $i=n+1$, $n+2$, and the 2-primary subgroup of $\pi_{n+2}(\vartheta_{n2})$ vanishes for $n\geqq 5$. Let S_n be regularly imbedded in ϑ_{np}. We show that every map f: $\partial E_{n+4}\rightarrow S_n$ which represents a 3-primary element e_f of $\pi_{n+3}(S_n)$, $n\geqq 3$, extends in ϑ_{n3} over E_{n+4} where E_{n+4} denotes an oriented $(n+4)$-cell. Actually, more results on this homotopy property of ϑ_{np} are obtained. For instance, we show that if F: $E_{n+4}\rightarrow\vartheta_{n3}$ is an extension of f above, then e_f is characterized by $F^*\xi^{-1}\mathcal{P}^1 g_n^*\in H^{n+4}(E_{n+4}, \partial E_{n+4}; Z_3)$ where Z_3 denotes the groups of integers mod 3 and ξ is the injection homomorphism: $H^{n+4}(\vartheta_{n3}, S_n; Z_3)\rightarrow H^{n+4}(\vartheta_{n3}, Z_3)$ which is an isomorphism. In the last section of this paper, we consider an $(n-1)$-connected finite complex K, $n\geqq 2$, and regularly imbed K into its 2-fold symmetric product. We show how this symmetric product is related to the problem of secondary obstructions of maps of a complex into K, a solution of which was originally given by N. E. Steenrod [10] and generalized by J. H. C. Whitehead [16].

I. Γ-PRODUCTS. GENERAL PROPERTIES

1. **Γ-products, homotopy type.** Let X be a locally compact, paracompact Hausdorff space, and X^q the q-fold Cartesian product of X. Let Γ be a group of permutations of the q letters $1, 2, \cdots, q$. We shall regard Γ as a transformation group acting on X^q in a natural fashion as follows: For any $\gamma\in\Gamma$ and $x=(x_1, x_2, \cdots, x_q)\in X^q$ we set $\gamma(x)=(x_{\gamma(1)}, x_{\gamma(2)}, \cdots, x_{\gamma(q)})$. The orbit space over X^q relative to Γ (obtained by identifying any two points x, x' of X^q into a single point whenever $x'=\gamma(x)$ for some $\gamma\in\Gamma$) will be denoted by X^Γ. X^Γ is a locally compact, paracompact Hausdorff space. It may be called the Γ-product of X. It is the q-fold cyclic product or the q-fold symmetric product of X when Γ is the cyclic group of order q or the symmetric group Σ_q of the q letters $1, 2, \cdots, q$. We shall write $\bar{I}$ for the identification map: $X^q\rightarrow X^\Gamma$. Let Y be also a locally compact, paracompact Hausdorff space and f: $X\rightarrow Y$ a map. We shall write f^q: $X^q\rightarrow Y^q$, and f^Γ: $X^\Gamma\rightarrow Y^\Gamma$ respectively for the q-fold Cartesian product of f and the map determined naturally by f such that $f^\Gamma\bar{I}=\bar{I}f^q$. For $q=2$, we write X^{Σ_2} as $X*X$ and write f^{Σ_2} as $f*f$(2).

We shall introduce a natural simplicial decomposition of the Cartesian product and hence of the Γ-product of a complex(3). The meaning of an ordered simplex and of an ordered simplicial complex will be understood as usual. Every simplicial complex can be ordered. We write an ordered s-simplex

(2) For known topological properties of cyclic and symmetric products, one may see [7, p. 184] and an announcement of S. K. B. Stein, Bull. Amer. Math. Soc. vol. 58 (1952) p. 207.

(3) The simplicial decomposition we shall introduce is simpler than that used by M. Richardson, *On the homology characters of symmetric products*, Duke Math. J. vol. 1 (1935) pp. 50–69.

By a complex we mean one of which a locally finite simplicial decomposition exists. Later, we shall consider also the *CW*-complexes of J. H. C. Whitehead [14, p. 223], which we shall refer to as cellular complexes.

σ as $\sigma=[w_0<w_1<\cdots<w_s]$ with vertices $w_0, w_1, \cdots, w_s$ and order $<$. Let $\sigma_k=[w_{0k}<w_{1k}<\cdots<w_{s_k k}]$ be given ordered simplexes, $k=1, 2, \cdots, q$. A simplicial decomposition of $\sigma^{\#}=\sigma_1\times\sigma_2\times\cdots\times\sigma_q$, denoted by $\bar{\sigma}^{\#}$, is as follows. The vertices of $\bar{\sigma}^{\#}$ are the points $\bar{w}=(w_{j_1 1}, w_{j_2 2}, \cdots, w_{j_q q})$ of $\bar{\sigma}^{\#}$. Write $\bar{w}<\bar{w}'$ for any distinct $\bar{w}=(w_{j_1 1}, w_{j_2 2}, \cdots, w_{j_q q})$, $\bar{w}'=(w_{j_1' 1}, w_{j_2' 2}, \cdots, w_{j_q' q})$ in case that $w_{j_k k}\leqq w_{j_k' k}$, $k=1, 2, \cdots, q$. The n-simplexes of $\bar{\sigma}^{\#}$ are those spanned by any $n+1$ points $\bar{w}_0<\bar{w}_1<\cdots<\bar{w}_n$([4]). Let K_k be ordered simplicial complexes, $k=1, 2, \cdots, q$. It is then clear that the union of all $\bar{\sigma}^{\#}$ where $\sigma^{\#}=\sigma_1\times\sigma_2\times\cdots\times\sigma_q$ with σ_k in K_k constitutes a simplicial decomposition of

$$K^{\#} = K_1 \times K_2 \times \cdots \times K_q,$$

which we shall denote by $\overline{K}^{\#}$.

It is easy to see that if K is an ordered simplicial complex, Γ acting on K^q leaves the simplicial decomposition $\overline{K}^q$ invariant. Moreover, if $\gamma\in\Gamma$ leaves a simplex $\bar{\tau}$ in $\overline{K}^q$ spanned by $\bar{w}_0<\bar{w}_1<\cdots<\bar{w}_n$ invariant, then, since $\bar{w}_i<\bar{w}_j$ implies $\gamma(\bar{w}_i)<\gamma(\bar{w}_j)$, γ leaves every $\bar{w}_i$ invariant and hence leaves $\bar{\tau}$ pointwise invariant. It follows therefore that the identification map $\overline{I}$: $K^q\rightarrow K^\Gamma$ carries $\overline{K}^q$ naturally to a simplicial decomposition $\overline{K}^\Gamma$ of K^Γ.

In the remaining part of this section, we shall give some remarks on the homotopy type of Γ-products.

(1.1) *If f, f': $X\rightarrow Y$ are homotopic maps between locally compact, paracompact Hausdorff spaces, then f^Γ and f'^Γ are homotopic.*

In fact, let F: $X\times\langle 0, 1\rangle\rightarrow Y$ be a homotopy connecting f and f' (namely, a map such that $F(x, 0)=f(x)$, $F(x, 1)=f'(x)$ for any $x\in X$). Denote by W the subset of $(X\times\langle 0, 1\rangle)^\Gamma$ of all points $\overline{I}(x_1\times t, x_2\times t, \cdots, x_q\times t)$, $x_i\in X$, $t\in\langle 0, 1\rangle$. We see then that W is homeomorphic to $X^\Gamma\times\langle 0, 1\rangle$ in a natural manner and that the partial map $F^\Gamma|W$: $W\rightarrow Y^\Gamma$ is a homotopy connecting f^Γ and f'^Γ.

(1.2) *If X and Y have the same homotopy type, then X^Γ and Y^Γ have the same homotopy type.*

In fact, let f: $X\rightarrow Y$ and g: $Y\rightarrow X$ be maps such that $gf\simeq\iota$ (=the identity map) in X and $fg\simeq\iota$ in Y. Then, $g^\Gamma f^\Gamma=(gf)^\Gamma\simeq\iota$ in X^Γ and $f^\Gamma g^\Gamma=(fg)^\Gamma\simeq\iota$ in Y^Γ by (1.1).

As an immediate consequence of (1.2), we have

(1.3) *If X is contractible, then X^Γ is contractible.*

We shall write K_m for the mth skeleton of any given simplicial complex K.

(1.4) *The Γ-product of an m-connected complex is m-connected* ($m\geqq 0$).

In fact, let K be an oriented simplicial complex with a deformation of K_m in K into a point. Then, we verify easily that $(\overline{K}^q)_m$ is contained in $(K_m)^q$ and hence that $(\overline{K}^q)_m$ is deformable in K^q into a point by a deformation in-

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([4]) This verification needs only some elementary treatment in analytical geometry.

variant under Γ. It follows that $\bar{I}$ carries this deformation to a deformation in K^Γ of the mth skeleton of $\bar{K}^\Gamma$ into a point. This proves (1.4).

2. **Special cohomology.** The cohomology theory over a locally compact, paracompact Hausdorff space W is given as usual, based on a complete system of star-finite (open) coverings of W. (For convenience, we shall allow the repetition of the same open set as different members of the covering.) Let Σ be a finite transformation group acting on W. We shall give some preliminaries on the cohomology arising from the fashion of Σ acting on W for later applications. Denote by $O(W, \Sigma)$ the orbit space over W relative to Σ and by $\bar{I}$ the identification map of W to the orbit space. (If Σ is a cyclic group with generator T, we shall write $O(W, \Sigma)$ as $O(W, T)$ in accordance with notation used in [3].) Denote also by $\Sigma^\#$ the integral group ring generated by the elements of Σ.

Let $\{V_\lambda\}$ be a complete system of star-finite coverings of W, invariant under Σ, such that, for each element $\gamma \in \Sigma$, and each member A of V_λ, either $\gamma(A)$ and A are disjoint or coincide. Let K_λ be the nerve of V_λ. Let W_i be a closed subset of W, invariant under Σ. We write $K_{i\lambda}$ for the subcomplex of K_λ consisting of all simplexes whose nucleus meets W_i. Σ acts then on K_λ in a natural manner, leaving $K_{i\lambda}$ invariant, and, due to the construction of $\{V_\lambda\}$, we see that a projection $\pi = \pi_{\mu\nu}$: $K_\mu \rightarrow K_\nu$ invariant under Σ can be chosen when V_μ refines V_ν.

Let $\phi \in \Sigma^\#$ and $\epsilon = \pm 1$. Let $W_i \supset W_j$ be closed subsets of W, invariant under Σ. ϕ gives then a cochain map ϕ in the group $C_s(K_{i\lambda}, K_{j\lambda}; G)$ of s-cochains of the pair $(K_{i\lambda}, K_{j\lambda})$ with coefficient group G. We shall write ${}^{\phi^\epsilon}C_s(K_{i\lambda}, K_{j\lambda}; G)$ as the image of this cochain map ϕ if $\epsilon = 1$ and as its kernel if $\epsilon = -1$. More generally, if α is a set of ϕ^ϵ's with $\phi \in \Sigma^\#$, $\epsilon = \pm 1$, we write ${}^\alpha C_s(K_{i\lambda}, K_{j\lambda}; G)$ as the intersection of the ${}^{\phi^\epsilon}C_s(K_{i\lambda}, K_{j\lambda}; G)$'s. The group ${}^\alpha C_s(K_{i\lambda}, K_{j\lambda}; G)$ for all s (and a fixed λ) form a Mayer cochain complex M_λ(5) and we may thus define the cohomology group of M_λ, to be denoted by ${}^\alpha H^s(K_{i\lambda}, K_{j\lambda}; G)$. We shall define the cohomology group ${}^\alpha H^s(W_i, W_j; G)$ as the direct limit in λ of ${}^\alpha H^s(K_{i\lambda}, K_{j\lambda}; G)$, using the projection π before. In a similar manner, we may define homology groups of this sort.

If $\alpha = \{\phi\}$ (i.e., α consists of one single element ϕ of $\Sigma^\#$), we write: ${}^\alpha H^s(W_i, W_j; G)$ as ${}^\phi H^s(W_i, W_j; G)$, and if $\alpha = \{\phi^{-1}\}$ we write ${}^\alpha H^s(W_i, W_j; G)$ as ${}^{\phi^{-1}}H^s(W_i; W_j; G)$. Let $W_1 \supset W_2 \supset W_3$ be a triple of closed subsets of W, invariant under Σ. We verify easily the commutativity in each square of the following diagram (2.1), in which one of the row sequences is the cohomology sequence of the triple $W_1 \supset W_2 \supset W_3$, other two row sequences are defined in a similar manner as for the cohomology sequences of a triple, and all the homomorphisms in the column sequences are defined in the same way as in [3, p. 74].

(5) See J. L. Kelley and E. Pitcher, *Exact homomorphism sequences in homology theory*, Ann. of Math. vol. 48 (1947) pp. 682–702.

(2.1)

$$
\begin{array}{ccccccccc}
 & \vdots & & \vdots & & \vdots & & \vdots & \\
 & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\cdots \to & {}^{\phi^{-1}}H^{s+1}(W_1, W_2; G) & \xrightarrow{\xi} & {}^{\phi^{-1}}H^{s+1}(W_1, W_3; G) & \xrightarrow{\eta} & {}^{\phi^{-1}}H^{s+1}(W_2, W_3; G) & \xrightarrow{\delta} & {}^{\phi^{-1}}H^{s+2}(W_1, W_2; G) & \to \cdots \\
 & \uparrow k_\phi & & \uparrow k_\phi & & \uparrow k_\phi & & \uparrow -k_\phi & \\
\cdots \to & {}^{\phi}H^{s}(W_1, W_2; G) & \xrightarrow{\xi} & {}^{\phi}H^{s}(W_1, W_3; G) & \xrightarrow{\eta} & {}^{\phi}H^{s}(W_2, W_3; G) & \xrightarrow{\delta} & {}^{\phi}H^{s+1}(W_1, W_2; G) & \to \cdots \\
 & \uparrow j_\phi & & \uparrow j_\phi & & \uparrow j_\phi & & \uparrow j_\phi & \\
\cdots \to & H^{s}(W_1, W_2; G) & \xrightarrow{\xi} & H^{s}(W_1, W_3; G) & \xrightarrow{\eta} & H^{s}(W_2, W_3; G) & \xrightarrow{\delta} & H^{s+1}(W_1, W_2; G) & \to \cdots \\
 & \uparrow i_\phi & & \uparrow i_\phi & & \uparrow i_\phi & & \uparrow i_\phi &
\end{array}
$$

All these row and column sequences are exact.

Let $\phi_1, \phi_2 \in \Sigma^{\#}$ be such that $\phi_1\phi_2 = 0$. Then, clearly, ${}^{\phi_2}C_s(K_{i\lambda}, K_{j\lambda}; G) \subset {}^{\phi_1^{-1}}C_s(K_{i\lambda}, K_{j\lambda}; G)$ for each λ, and these inclusions give naturally a homomorphism

$$\lambda: {}^{\phi_2}H^s(W_i, W_j; G) \to {}^{\phi_1^{-1}}H^s(W_i, W_j; G). \tag{2.2}$$

We verify that, with ξ, η, δ, λ in (2.1) and (2.2),

$$\lambda\xi = \xi\lambda, \qquad \lambda\eta = \eta\lambda, \qquad \lambda\delta = \delta\lambda. \tag{2.3}$$

3. **The isomorphism μ, regular imbedding.** We shall study a natural imbedding of X into X^Γ and some relationship between cohomology groups of X and of X^Γ.

Consider again the space W and the transformation group Σ in §2. We retain the usage of the notations there. When α consists of all $(1-a)^{-1}$ with $a \in \Sigma$, we write ${}^{\alpha}H^s(W_i, W_j; G)$ as ${}^{\Sigma^{-1}}H^s(W_i, W_j; G)$. We shall define a natural isomorphism

$$\omega: H^s(O(W_i, \Sigma), O(W_j, \Sigma); G) \approx {}^{\Sigma^{-1}}H^s(W_i, W_j; G). \tag{3.1}$$

In fact, it is clear that the identification map $\bar{I}: W \to O(W, \Sigma)$ carries $\{V_\lambda\}$ into a complete system of coverings $\{\bar{I}(V_\lambda)\}$ of $O(W, \Sigma)$, and for each λ, $O(K_\lambda, \Sigma)$ is the nerve $\bar{I}(V_\lambda)$. Also, the identification map $K_\lambda \to O(K_\lambda, \Sigma)$ induces isomorphism: $H^s(O(K_{i\lambda}, \Sigma), O(K_{j\lambda}, \Sigma); G) \approx {}^{\Sigma^{-1}}H^s(K_i, K_j; G)$. Taking direct limit in λ by the projection π gives the isomorphism (3.1).

Now, let X be connected. We shall define for each s an into-isomorphism

$$\mu: H^s(X, G) \to H^s(X^\Gamma, G). \tag{3.2}$$

In fact, if $s = 0$, both the groups in (3.2) are isomorphic to G in a natural manner, and we define μ in a trivial way. We assume in the following that $s > 0$. Let $\{V_\lambda\}$ be the system of coverings of X^q invariant under the symmetric group Σ_q of the q letters $1, 2, \cdots, q$, as given previously in §2 (for W and Σ). Let us write Σ'_{q-1} for the subgroup of Σ_q of permutations which leaves 1 fixed, and write e for the unit of the integral cohomology ring of an arbitrary locally compact, paracompact, connected Hausdorff space. If

$v \in H^s(X, G)$, then the isomorphism ω in (3.1) for $\Sigma = \Sigma'_{q-1}$ and $(W_i, W_j) = (X^q, 0)$ carries $v \times e \in H^s(X \times X^{\Sigma_{q-1}'}, G)$ into an element of ${}^{\Sigma_{q-1}'^{-1}}H^s(X^q, G)$. It follows that $v \times e \times \cdots \times e$ (q factors) $\in H^s(X^q, G)$ are represented by cocycles $z_{\lambda'}$, determined by this isomorphism, over the nerves of some coverings $V_{\lambda'}$ in $\{V_\lambda\}$, and invariant under Σ'_{q-1}. Thus, if T is a cyclic permutation of order q in Σ_q, then $\sum_{i=0}^{q-1} T^i(z_{\lambda'})$ is invariant under Γ. Write $\theta(v)$ as the element of ${}^{\Gamma^{-1}}H^s(X^q, G)$ represented by $\sum_{i=0}^{q-1} T^i(z_{\lambda'})$. $\theta(v)$ is then determined independently of the choice of $z_{\lambda'}$, and $\theta: H^s(X, G) \to {}^{\Gamma^{-1}}H^s(X^q, G)$ is a homomorphism. Let us define

$$\mu = \omega^{-1}\theta: H^s(X, G) \to H^s(X^\Gamma, G)$$

where ω is the isomorphism (3.1) for $\Sigma = \Gamma$ and $(W_i, W_j) = (X^q, 0)$.

We assert that μ is an into-isomorphism. To see this, we associate $\theta'(v) = \sum_{i=0}^{q-1} T^i(v \times e \times \cdots \times e) \in H^s(X^q, G)$ to each $v \in H^s(X, G)$ with T given above. Clearly, θ' is an into-isomorphism: $H^s(X, G) \to H^s(X^q, G)$. By the definition of ω, the commutativity of the diagram

(3.3)
$$\begin{array}{ccc} & & H^s(X^\Gamma, G) \\ & \swarrow \bar{I}^* & \uparrow \mu \\ H^s(X^q, G) & \xleftarrow[\theta']{} & H^s(X, G) \end{array}$$

holds. Therefore, μ is an into-isomorphism.

We verify easily also the following statements (3.4) and (3.5) by the definition of μ.

(3.4) *Let $f: X \to Y$ be a map between locally compact, paracompact connected Hausdorff spaces. The commutativity of the diagram*

$$\begin{array}{ccc} H^s(Y, G) & \xrightarrow{\mu} & H^s(Y^\Gamma, G) \\ \downarrow f^* & & \downarrow (f^\Gamma)^* \\ H^s(X, G) & \xrightarrow{\mu} & H^s(X^\Gamma, G) \end{array}$$

holds. The subgroup $\mu(H^s(X, G))$ of $H^s(X^\Gamma, G)$ is invariant under homeomorphisms of X.

Let Γ' be a subgroup of Γ. Then, every orbit over X^q relative to Γ' is contained in a certain orbit relative to Γ and this gives a natural map $\eta_{\Gamma'\Gamma}: X^{\Gamma'} \to X^\Gamma$.

(3.5) *If Γ' is a subgroup of Γ, the commutativity of the diagram*

$$\begin{array}{ccc} & & H^s(X, G) \\ & \swarrow \mu & \downarrow \mu \\ H^s(X^{\Gamma'}, G) & \xleftarrow[\eta^*_{\Gamma'\Gamma}]{} & H^s(X^\Gamma, G) \end{array}$$

holds.

The terminology "regularly imbedding of X into X^{Γ}" is used in the following sense. Let s_0 be an arbitrary point of X. We write

$$s_0(X) = \bar{I}(X \times s_0 \times \cdots \times s_0) \in X^{\Gamma}.$$

Let $f_{s_0}\colon X \to X^q$ be the map which sends $x \in X$ to $(x, s_0, \cdots, s_0) \in X^q$. Let $f'_{s_0} = \bar{I} f_{s_0}$. It is (regarded as) a homeomorphism of X onto $s_0(X)$. Identifying X and $s_0(X)$ under this homeomorphism, we say that X is regularly imbedded in X^{Γ} with reference point s_0. The commutativity of the diagram

$$\begin{array}{ccc} H^s(X^{\Gamma}, G) & \xrightarrow{\eta} & H^s(s_0(X), G) \\ \downarrow \bar{I}^* & \nwarrow \mu & \downarrow f'^*_{s_0} \\ H^s(X^q, G) & \xrightarrow[f^*_{s_0}]{} & H^s(X, G) \end{array}$$

holds, where the upper η is the homomorphism induced by the inclusion $s_0(X) \subset X^{\Gamma}$. If $s > 0$, for any $v \in H^s(X, G)$, $f^*_{s_0}$ maps $\theta'(v)$ to v so that, by (3.3),

$$\eta\colon \mu(H^s(X, G)) \approx H^s(s_0(X), G).$$

If $s = 0$, this isomorphism is obviously true. Finally, let us state

(3.6) *If X is regularly imbedded in X^{Γ}, then $\eta\mu =$ the identity isomorphism of $H^s(X, G)$ for all s.*

II. The reduced power operation $\mathcal{P}^k$ in cyclic products of spheres

4. **Remarks on periodic maps.** Let W be a finite-dimensional compact Hausdorff space and T a periodic map of W of prime period p with fixed point set F. We shall give some remarks on the cohomology arising from the fashion of T acting on W for uses in later §§5, 7. It is assumed that the reader is somewhat familiar with the special homology theory on periodic maps [3; 8]. In general, we shall use a theory on W module a closed subset of W invariant under T. This generalization to the relative case can be easily obtained in a way parallel to that in [3]. For necessary definitions of relative special cohomology (and homology) groups, we may see §2.

We shall write Z and Z_p respectively for the groups of integers and the group of integers mod p. The notations σ, τ, ρ, $\bar{\rho}$, etc. will be referred to [3]. We denote by W_0, W_1, W_2, W_3 nonempty closed subsets of W invariant under T and by E_n an n-cell with boundary sphere S_{n-1}.

(4.1) *With W, T, F, W_0 given above, let the pair (W, W_0) have the same cohomology groups* mod p *as the pair (E_n, S_{n-1}). Then:*

(i) *$H^s(F, W_0 \cap F; Z_p) \approx Z_p$ for $s =$ a certain $r \geqq 0$ and vanishes for other s.*

(ii) *$H^s(O(W, T), O(W_0, T); Z_p) \approx Z_p$ for $r+1 < s \leqq n$ and vanishes for other s in case $r < n$; $H^s(O(W, T), O(W_0, T); Z_p) \approx Z_p$ for $s = r$ and vanishes for other s in case $r = n$.*

(4.1) is a generalization to the relative case of results in [3, p. 78] on periodic maps acting on a homology sphere. It can be proved in a way parallel to that for the absolute case. The detailed verification is thus omitted.

Let (W, W_0) have the same integral cohomology groups as (E_n, S_{n-1}). Then, (W, W_0) has the same cohomology groups mod p as (E_n, S_{n-1}) by the usual relationship between cohomology groups given by the coefficient sequence $0 \to Z \to Z \to Z/pZ \to 0$ [3, pp. 71–73]. It is easy to see that there is one integer $d(T)$ which is ± 1 such that $T^*(v) = d(T)v$ for all $v \in H^n(W, W_0; Z)$. If $d(T) = -1$, p must be 2.

(4.2) *With W, W_0, T given above such that (W, W_0) has the same integral cohomology groups as (E_n, S_{n-1}), and with r given in* (4.1), *we have: $n-r$ is even or odd according as $d(T)=1$ or $=-1$. Moreover,*

(i) *If $d(T)=1$, $H^s(O(W, T), O(W_0, T); Z) \approx Z_p$ for $s=r+3, r+5, \cdots, n-1$; $\approx Z$ for $s=n$ and vanishes for other s.*

(ii) *If $d(T)=-1$, $H^s(O(W, T), O(W_0, T); Z) \approx Z_p$ for $s=r+3, r+5, \cdots, n$ and vanishes for other s.*

This is the generalization to the relative case of results in [3, p. 82].

(4.3) *With W, W_1, W_2 and T given before, let $W_1 \supset W_2$ and $i^*: H^s(W, W_1; Z_p) \approx H^s(W, W_2; Z_p)$ for all s where i denotes the inclusion $(W, W_2) \subset (W, W_1)$. Then, $\bar{\imath}^*: H^s(O(W, T), O(W_1, T); Z_p) \approx H^s(O(W, T), O(W_2, T); Z_p)$ for all s where $\bar{\imath}$ denotes the inclusion $(O(W, T), O(W_2, T)) \subset (O(W, T), O(W_1, T))$.*

By exactness property of cohomology sequence, it suffices to show that $H^s(O(W_1, T), O(W_2, T); Z_p) = 0$ for all s, using that $H^s(W_1, W_2; Z_p) = 0$ for all s. Using arguments as in [3, pp. 74–77], it is easy to see that if W has trivial cohomology groups mod p, then $O(W, T)$ has trivial cohomology groups mod p. Generalization of this to the relative case yields (4.3).

(4.4) *With W_1, W_2, W_3, T given before, suppose that* (i) *$W_1 \supset W_2 \supset W_3$,* (ii) *$(W_i, W_{i+1})$ and (E_{n_i}, S_{n_i-1}) have the same cohomology groups* mod p *such that the integer r_i given as in* (4.1) *corresponding to (W_i, W_{i+1}) is less than $n_i - 1$, $i=1, 2$, and that $n_2 < n_1$, and* (iii) *the coboundary homomorphism: $H^{n_2}(W_2, W_3; Z_p) \to H^{n_2+1}(W_1, W_2; Z_p)$ is trivial. Then, the coboundary homomorphism: $H^s(O(W_2, T), O(W_3, T); Z_p) \to H^{s+1}(O(W_1, T), O(W_2, T); Z_p)$ is trivial for all s.*

In fact, using arguments as in [3, pp. 75–77], ${}^{\rho}H^s(W_i, W_{i+1}; Z_p) \approx Z_p$ for $r_i < s \leqq n$ and vanishes for other s, $i=1, 2$, and

$$\lambda: {}^{\rho}H^s(W_i, W_{i+1}; Z_p) \approx {}^{\bar{\rho}-1}H^s(W_i, W_{i+1}; Z_p) \quad \text{for } r_i + 1 < s \leqq n_i,\ i = 1, 2, \tag{4.5}$$

where λ is the homomorphism (2.2). We consider the diagram (2.1), and we have, in the row exact sequences in this diagram,

$$j_\rho: H^{n_i}(W_i, W_{i+1}; Z_p) \approx {}^{\rho}H^{n_i}(W_i, W_{i+1}; Z_p), \tag{4.6}$$

$$k_\rho: {}^{\rho}H^s(W_i, W_{i+1}; Z_p) \approx {}^{\rho-1}H^{s+1}(W_i, W_{i+1}; Z_p), \quad s \neq n_i, \tag{4.7}$$

$i = 1, 2.$

By the hypothesis (iii), (4.6), and the commutativity property of the diagram (2.1), we get at once that δ: ${}^{\rho}H^{n_2}(W_2, W_3; Z_p) \to {}^{\rho}H^{n_2+1}(W_1, W_2; Z_p)$ is a trivial homomorphism, so that, by (4.5) and (2.3), δ: ${}^{\bar{\rho}^{-1}}H^{n_2}(W_2, W_3; Z_p) \to {}^{\bar{\rho}^{-1}}H^{n_2+1}(W_1, W_2; Z_p)$ is a trivial homomorphism. Again, using this, (4.7), and the commutativity of (2.1), we get that δ: ${}^{\rho}H^{n_2-1}(W_2, W_3; Z_p) \to {}^{\rho}H^{n_2}(W_1, W_2; Z_p)$ is a trivial homomorphism. Continuing in this way with the aid of the hypothesis $n_2 < n_1$, we arrive at δ: ${}^{\bar{\rho}^{-1}}H^{s}(W_2, W_3; Z_p) \to {}^{\bar{\rho}^{-1}}H^{s+1}(W_1, W_2; Z_p)$ is trivial for all s. Taking $\rho = \tau$ and using the isomorphism (3.1) we complete the proof of (4.4).

5. **Cyclic products of spheres.** We shall write

$$\vartheta_{np} = \text{the } p\text{-fold cyclic product of an } n\text{-sphere } S_n,$$

and denote by T the periodic map used in the construction of ϑ_{np}. Only the case p = prime will be considered in this paper. We shall calculate the integral cohomology groups of ϑ_{np} $(n \geqq 2)$ in this section. We denote by J throughout the group Z or the group Z_p.

We decompose S_n into a cellular complex composed of a point s_0 and an open cell C_n and consider the p-fold Cartesian product ϑ^0_{np} of S_n and the following subsets of ϑ^0_{np}, ϑ_{np}:

$$\overset{0}{\Gamma}_{np} = \overset{0}{\vartheta}_{np} - \overset{0}{\Delta}_{np} \quad \text{where} \quad \overset{0}{\Delta}_{np} = \text{the } p\text{-fold Cartesian product of } C_n,$$
$$\Gamma_{np} = \bar{I}(\overset{0}{\Gamma}_{np}).$$

Since the fixed point set of T is the diagonal sphere of ϑ^0_{np}, which meets Γ^0_{np} at one point, namely, the p-fold Cartesian product s_0^p of s_0, $T(\Gamma^0_{np}) = \Gamma^0_{np}$ and $H^s(\vartheta^0_{np}, \Gamma^0_{np}; J) \approx J$ for $s = np$ and vanishes for other s, we have therefore by (4.1) and (4.2) that

$$(5.1) \qquad H^s(\vartheta_{np}, \Gamma_{np}; Z) \approx \begin{cases} Z & \\ Z_p & \text{for} \\ Z_p & \\ 0 & \end{cases} \begin{array}{l} s = np \text{ with } np - n = \text{even}, \\ s = np \text{ with } np - n = \text{odd}, \\ s = n + 2k + 1 \leqq np \text{ with } k > 0, \\ \text{other } s, \end{array}$$

$$(5.2) \qquad H^s(\vartheta_{np}, \Gamma_{np}; Z_p) \approx \begin{cases} Z_p & \text{for} \\ 0 & \end{cases} \begin{array}{l} s = 0 \text{ and } n + 2 \leqq s \leqq np, \\ \text{other } s. \end{array}$$

The given cellular decomposition of S_n gives naturally a cellular decomposition of Γ^0_{np} composed of all p-fold products $B = B_1 \times B_2 \times \cdots \times B_p$ where each B_i denotes either s_0 or C_n and not all B_i are C_n, and this yields a cellular decomposition of Γ_{np} with cells of the form $\bar{I}(B)$, since T leaves only the points in the diagonal sphere of ϑ^0_{np} fixed. In this cellular decomposition of Γ_{np}, there is no s-cell if s is not a multiple nr of n with $0 \leqq nr \leqq n(p-1)$ and there are $C_{p,r}/p$ s-cells if $0 < s = nr \leqq n(p-1)$ (where $C_{p,r}$ is the binomial coefficient). It is therefore easy to see that

$$(5.3)\qquad H^s(\Gamma_{np}, J) \approx \begin{cases} J & s = 0, \\ \{J, C_{p,r}/p\} & \text{for } 0 < s = nr \leqq n(p-1),\ n \geqq 2, \\ 0 & \text{other } s, \end{cases}$$

where $\{J, t\}$ denotes the direct sum of t groups each of which is isomorphic to J.

(5.4) *The sequence*

$$(5.5)\qquad 0 \to H^s(\vartheta_{np}, \Gamma_{np}; J) \xrightarrow{\xi} H^s(\vartheta_{np}, J) \xrightarrow{\eta} H^s(\Gamma_{np}, J) \to 0, \qquad n \geqq 2,$$

is exact(6), *where* ξ, η *are the injection and projection homomorphisms in the cohomology sequence of the pair* $(\vartheta_{np}, \Gamma_{np})$. *Then,*

$$H^s(\vartheta_{np}, J) \approx H^s(\vartheta_{np}, \Gamma_{np}; J) \oplus H^s(\Gamma_{np}, J), \qquad n \geqq 2,$$

with $H^s(\vartheta_{np}, \Gamma_{np}; J)$, $H^s(\Gamma_{np}, J)$ *given in* (5.1)–(5.3).

We need prove that

$$(5.6)\qquad \eta: H^s(\vartheta_{np}, J) \to H^s(\Gamma_{np}, J)$$

is onto for every s. This is obvious if s is not a multiple nr of n with $0<nr \leqq n(p-1)$. Let now $0<s=nr\leqq n(p-1)$. Since T acting on Γ^0_{np} leaves only one point, namely s_0^p, fixed, using a simplicial decomposition of ϑ^0_{np} as described in §1 with s_0^p as a vertex, we verify straightforwardly that ${}^\sigma H^s(\Gamma^0_{np}, Z_p) = {}^{\tau^{-1}}H^s(\Gamma^0_{np}, Z_p) \approx \{Z_p, C_{p,r}/p\}$ and that the homomorphism j_σ: $H^s(\Gamma^0_{np}, Z_p) \to {}^\sigma H^s(\Gamma^0_{np}, Z_p)$ is onto with j_σ given as in (2.1). Clearly, the projection homomorphism η: $H^s(\vartheta^0_{np}, Z_p) \to H^s(\Gamma^0_{np}, Z_p)$ is onto. Thus, by the commutativity property of (2.1), we have that η: ${}^\sigma H^s(\vartheta^0_{np}, Z_p) \to {}^\sigma H^s(\Gamma^0_{np}, Z_p)$ is onto with η given as in (2.1). (5.6) for $J=Z_p$ follows now easily from (2.3) and the isomorphism (3.1). (5.6) for $J=Z$ follows from this and some usual relationship between cohomology groups given by the coefficient sequence $0\to Z \to Z\to Z/pZ\to 0$(7).

We shall write

$$(5.7)\qquad \begin{aligned} Q^s(\vartheta_{np}, J) &= \text{the kernel of } \eta \text{ in (5.5)}, \\ \overset{*}{g}_n &= \text{a generator of the infinite cyclic group } H^n(\vartheta_{np}, Z), \\ \overset{\#}{g}_n &= \overset{*}{g}_n \bmod p, \text{ which generates } H^n(\vartheta_{np}, Z_p). \end{aligned}$$

(5.8) *Let n be even. Let* e^0_{np}, e_{np} *be respectively generators of the infinite cyclic groups* $H^{np}(\vartheta^0_{np}, Z)$, $H^{np}(\vartheta_{np}, Z)$. *We have* $\bar{I}^*(e_{np}) = \pm p e^0_{np}$.

In fact, let us decompose ϑ^0_{np} into a simplicial complex K as in §1. The identification map $\bar{I}$: $\vartheta^0_{np}\to\vartheta_{np}$ carries K into a simplicial decomposition of

(6) For $n=1$, this sequence is not exact in general. An example is ϑ_{12}, which is the Möbius band.

(7) Here, the assumption $n\geqq 2$ is needed.

ϑ_{np}. ϑ_{np} being a pseudo-manifold, e_{np} is represented by an oriented np-simplex τ, and there are exactly p oriented np-simplexes in K which $\bar{I}$ maps into τ. Since n is even, it is easy to see that T preserves orientation of the manifold ϑ_{np}^0. This gives (5.8).

6. **The cyclic reduced powers of $g_n^\#$.** We assume $n \geqq 2$. Let S_n be an oriented n-sphere. Let e_0 be the unit of the integral cohomology ring of S_n and e_n the generator of $H^n(S_n, Z)$ represented by S_n. Write $e_{n,i}^0 = e_0 \times \cdots \times e_0 \times e_n \times \cdots \times e_0 \in H^n(\vartheta_{np}^0, Z)$ where the ith factor is e_n and all other factors are e_0. Then, $e_{np}^0 = \pm \prod_{i=1}^p e_{n,i}^0$ with e_{np}^0 given in (5.8) and a direct computation gives that, in ϑ_{np}^0, $(\sum_{i=1}^0 e_{n,i}^0)^p = 0$ or $= \pm p!\, e_{np}^0$ according as n is odd or even. Thus, by some properties of regular imbedding S_n in ϑ_{np} as given in §3, we see that $\bar{I}^*(g_n^*) = \pm \sum_{i=1}^p e_{n,i}^0$ so that, using (5.8),

$$(6.1) \qquad g_n^{*p} = \pm (p-1)!\, e_{np} \quad \text{and hence} \quad g_n^{\# p} \neq 0, \qquad n = \text{even}.$$

Following [9], $\mathcal{P}^k$ will be used to denote the cyclic reduced power operation mod p (and for $p=2$, $\mathcal{P}^k$ means the square operation Sq^{2k}). We have, in ϑ_{np},

$$(6.2) \qquad \mathcal{P}^k g_n^\# \neq 0, \qquad 2k \leqq n.$$

Although this follows also from some general considerations in later sections, we may prove it, in a simple way, as follows. Let $n=2$. Then, $\mathcal{P}^0 g_2^\# = g_2^\# \neq 0$ and $\mathcal{P}^1 g_2^\# = g_2^{\# p} \neq 0$ by (6.1). We shall complete the proof of (6.3) by induction on n. Let E_{n+1} be an $(n+1)$-cell(8) with boundary S_n, and let $f\colon E_{n+1} \to S_{n+1}$ be the map which shrinks S_n into a single point s of S_{n+1} and is topological elsewhere. Denote by ϑ'_{n+1p} the p-fold cyclic product of E_{n+1}. Then $\vartheta_{np} \subset \vartheta'_{n+1p}$, and f gives rise naturally to a map $F\colon \vartheta'_{n+1p} \to \vartheta_{n+1p}$ such that $F\bar{I}(x) = \bar{I}f^p(x)$ where x is an arbitrary point of the p-fold Cartesian product of E_{n+1}. Since ϑ'_{n+1p} is contractible by (1.3), the coboundary homomorphism

$$(6.3) \qquad \delta\colon H^s(\vartheta_{np}, Z_p) \to H^{s+1}(\vartheta'_{n+1p}, \vartheta_{np}; Z_p), \qquad s > 0,$$

is an into-isomorphism. Also, it is clear that $F(\vartheta_{np}) = \bar{I}(s^p) \in \vartheta_{n+1p}$ (with s^p = the p-fold Cartesian product of $s \in S_{n+1}$). Thus, F induces the homomorphism

$$F^*\colon H^s(\vartheta_{n+1p}, (s^p); Z_p) \to H^s(\vartheta'_{n+1p}, \vartheta_{np}; Z_p).$$

We assert that F^* is an onto-isomorphism for $s = n+1$. In fact, we have the into-homeomorphisms $f_s\colon S_{n+1} \to \vartheta_{n+1p}$ $(s \in S_{n+1})$, $f_{s_0}\colon E_{n+1} \to \vartheta'_{n+1p}$ $(s_0 \in S_n)$ as given in the last paragraph of §3. We verify easily that $F(f_{s_0}(S_n)) = s^p$ and F maps $f_{s_0}(E_{n+1} - S_n)$ topologically onto $f_s(S_{n+1} - (s))$. Then, by some property of regular imbedding given in §3, the isomorphism (6.3) and some usual

(8) By a cell we mean a space which is homeomorphic to a bounded closed convex cell in a Euclidean space.

arguments in cohomology theory, we prove our assertion.

Write $g'^{\#}_{n+1}$ for the generator of $H^{n+1}(\vartheta_{n+1p}, (s^p); Z_p)$ such that $\xi g'^{\#}_{n+1} = g^{\#}_{n+1}$ where ξ is the injection homomorphism: $H^s(\vartheta_{n+1p}, (s^p); Z_p) \to H^s(\vartheta_{n+1p}, Z_p)$. Then, if $2k \leqq n$, $f^{p*}\mathcal{P}^k g'^{\#}_{n+1} = \lambda\delta\mathcal{P}^k g^{\#}_n$ with $\lambda \in Z_p$, $\neq 0$ by our assertion above, and hence $\mathcal{P}^k g'^{\#}_{n+1} \neq 0$ by the isomorphism (6.3) and induction hypothesis. It follows that $\mathcal{P}^k g^{\#}_{n+1} \neq 0$. If $2k = n+1$, then $\mathcal{P}^k g^{\#}_{n+1} = g^{\#p}_{n+1} \neq 0$ by (6.1). This proves (6.2).

We assert also that

$$\mathcal{P}^k g^{\#}_n \in Q^{n+2k(p-1)}(\vartheta_{np}, Z_p), \qquad k > 0, \tag{6.4}$$

with $Q^{n+2k(p-1)}(\vartheta_{np}, Z_p)$ given in (5.7), and by (5.4), it remains to show that $\eta\mathcal{P}^k g^{\#}_n = 0$ for the case $n < n + 2k(p-1) = nr \leqq n(p-1)$. Write $d_{nr} \in H^{nr}(\Gamma_{np}, Z_p)$ for the element represented by $\sum_{i=1}^{q} C^i_{nr}$ where $C^1_{nr}, C^2_{nr}, \cdots, C^q_{nr}$, $q = C_{p,r}/p$, are the oriented open nr-cells in Γ_{np} with respect to the cellular decomposition of Γ_{np} given in §5 and with the orientation induced by the given orientation of S_n. In the r-fold Cartesian product ϑ^0_{nr} of S_n, let us denote by $e^0_{n,i}$ the element of $H^n(\vartheta^0_{nr}, Z_p)$ as given in the first paragraph of this section but with Z replaced by Z_p and with p replaced by r. Since for each j, $j = 1, 2, \cdots, q$, we may construct a map $T_j: \vartheta^0_{nr} \to \Gamma_{np}$ such that $T^*_j(C^j_{nr})$ represents $\prod_{i=1}^{r} e^0_{n,i}$ and $T^*_j(S_n)$ represents $\sum_{i=1}^{r} e^0_{n,i}$, and since in ϑ^0_{nr} we have clearly $\mathcal{P}^k T^*_j(S_n) = 0$, it follows therefore easily that $\eta\mathcal{P}^k g^{\#}_n = 0$. We get thus (6.4).

We shall write S^0_n, S^1_n respectively for the diagonal spheres in ϑ^0_{np} and $\bar{I}(S^0_n)$ in ϑ_{np} and write

$$\begin{aligned} &w^*_n \in H^n(S^1_n, Z) && \text{for the generator,} \\ &w_n = w^*_n \pmod p \in H^n(S^1_n, Z_p). \end{aligned} \tag{6.5}$$

By a consideration on the map $\bar{I}: \vartheta^0_{np} \to \vartheta_{np}$, we see easily that

$$\zeta g^*_n = \pm p w^*_n, \qquad \zeta g^{\#}_n = 0, \tag{6.6}$$

where ζ denotes the projection: $H^n(\vartheta_{np}, J) \to H^n(S^1_n, J)$.

7. **Neighbourhood of the diagonal sphere S^1_n in ϑ_{np}.** We intend to calculate the iterated cyclic powers of $g^{\#}_n$. We recall the following: Let $\Delta_{(m,p)}$ be the lens space which is the orbit space over an m-sphere S_m, $m \geqq 2$, relative to an orthogonal transformation in S_m of prime period p without fixed point. The cohomology groups $H^s(\Delta_{(m,p)}, Z_p)$ of $\Delta_{(m,p)}$ mod p are isomorphic to Z_p for $0 \leqq s \leqq m$. Let $v_1 \in H^1(\Delta_{(m,p)}, Z_p)$, $v_2 \in H^2(\Delta_{(m,p)}, Z_p)$ be generators. Then, v^k_2 generates $H^{2k}(\Delta_{(m,p)}, Z_p)$ for $0 \leqq 2k \leqq m$, $v_1 v^k_2$ generates $H^{2k+1}(\Delta_{(m,p)}, Z_p)$ for $0 \leqq 2k+1 \leqq m$, $v^2_1 = 0$, or $= v^2_2$ according as $p > 2$ or $= 2$ [17].

We may take S_n as an analytical manifold. Then, naturally, T acts on ϑ^0_{np} and S^0_n imbeds in ϑ^0_{np} analytically. We may then consider the fiber bundle $\mathcal{B}_0$

over S_n^0 whose fibers are $(np-n-1)$-dimensional normal spheres of S_n^0 in ϑ_{np}^0. Let E_0, ϕ_0, V_{0x} be the total space, the projection, and the fiber over $x\in S_n^0$ of this bundle. Let V'_{0x}, $x\in S_n^0$, be the solid $(np-n)$-sphere composed of the geodesics joining x to points of V_{0x}, and let $C_0=\bigcup_{x\in S_n^0}V'_{0x}$, which is the map cylinder of ϕ_0. The following are easily seen: T leaves E_0, C_0, V_{0x}, V'_{0x} invariant, T applied on V_{0x} is equivalent to an orthogonal transformation without fixed point. Thus, $\bar{I}$ carries $\mathcal{B}_0$ into a bundle $\mathcal{B}$ over S'_n with the lens space $\Delta_{(np-n-1,p)}$ as fiber. Write $E=\bar{I}(E_0)$ and $C=\bar{I}(C_0)$. $C-E$ is an open subset of ϑ_{np} containing S_n^1 and S_n^1 is a deformation retract of C.

We shall evaluate the cohomology groups $H^s(E, Z_p)$. To do this, we have to use arguments in the theory of periodic maps, although $\mathcal{B}$ is a bundle so simple as over a sphere. Let $x_0\in S_n^0$ be a given point. For simplicity, write

$$V_0 = V_{0x_0}, \qquad V'_0 = V'_{0x_0}, \qquad V = \bar{I}(V_0), \qquad V' = \bar{I}(V'_0).$$

Using the theorem of Feldbau, we see easily that $C_0-E_0-V'_0$ is an open np-cell, and hence $H^s(C_0, E_0\cup V'_0; Z_p)\approx Z_p$ for $s=np$ and vanishes for other s. Thus, since T leaves C_0 and $E_0\cup V'_0$ invariant, it follows from (4.1) that $H^s(C, E\cup V'; Z_p)\approx Z_p$ for $n+2\leqq s\leqq np$ and vanishes for other s. Applying similar arguments to the pair (V'_0, V_0), we find that $H^s(V'\cup E, E; Z_p)$ $\approx H^s(V', V; Z_p)\approx Z_p$ for $2\leqq s\leqq np-n$ and vanishes for other s. Now, clearly, the coboundary homomorphism: $H^s(V'_0\cup E_0, E_0; Z_p)\rightarrow H^{s+1}(C_0, V'_0\cup E_0; Z_p)$ is trivial. Thus, using (4.4) we have the exact sequence

$$(7.1)\qquad 0\rightarrow H^s(C, V'\cup E; Z_p)\xrightarrow{\xi} H^s(C, E; Z_p)\xrightarrow{\eta} H^s(V'\cup E, E; Z_p)\rightarrow 0,$$

where ξ and η denote respectively the injection and projection homomorphism in the cohomology sequence of the triple $E\subset E\cup V'\subset C$. Therefore,

$$(7.2)\qquad H^s(C, E; Z_p)\approx\begin{cases} Z_p & 2\leqq s\leqq n+1,\ np-n+1\leqq s\leqq np,\\ Z_p\oplus Z_p & \text{for } n+2\leqq s\leqq np-n,\\ 0 & \text{other } s.\end{cases}$$

We shall write

$$(7.3)\qquad Q^s(C, E; Z_p) = \text{the kernel of } \eta \text{ in (7.1)}.$$

Let

$$(7.4)\qquad \bar{w}_n = \phi^*(w_n)\in H^n(E, Z_p)$$

where w_n is given in (6.5) and ϕ denotes the projection of the bundle $\mathcal{B}$. We assert that $\bar{w}_n\neq 0$. In fact, if $p>2$, then the cross section of $\mathcal{B}_0$ and hence the cross section of $\mathcal{B}$ exists since $np-n-1\geqq n$; our assertion is thus obvious. Assume $p=2$. We observe that the intersection number mod 2 in ϑ_{n2}^0 of the fundamental n-cycle of S_n^0 with itself is 0. If $a_n\in H^n(\vartheta_{n2}^0, Z_2)$ is dual to this fundamental cycle, then $a_n^2=0$; it follows that the fundamental characteristic class mod 2 of $\mathcal{B}$ vanishes [13]. This proves that ϕ_0^*: $H^s(S_n^0, Z_2) \to H^s(E_0, Z_2)$ is an into-isomorphism. Our assertion follows easily. Now, since S_n^1 is a deformation retract of C, by (7.2) and the exactness of the cohomology sequence of the pair (C, E), we have that

$$(7.5)\qquad H^s(E, Z_p) \approx \begin{cases} Z_p & 0 \leqq s \leqq n-1,\ np-n \leqq s \leqq np-1, \\ Z_p \oplus Z_p \quad \text{for} & n \leqq s \leqq np-n-1, \\ 0 & \text{other } s. \end{cases}$$

Using (7.5), the cohomology ring mod p of a lens space, and Wang sequence of the bundle $\mathcal{B}$ [6, p. 471], namely,

$$(7.6)\qquad \begin{aligned} \cdots &\xrightarrow{\alpha} H^s(E, Z_p) \xrightarrow{\beta} H^s(\Delta_{(np-n-1,p)}, Z_p) \\ &\xrightarrow{d_n} H^n(S_n^1, Z_p) \otimes H^{s-n+1}(\Delta_{(np-n-1,p)}, Z_p) \\ &\xrightarrow{\alpha} H^{s+1}(E, Z_p) \xrightarrow{\beta} \cdots, \end{aligned}$$

we can find easily the cohomology ring of E mod p. Notice that the homomorphism β in (7.6) may be interpreted geometrically as the projection of the cohomology groups of E into those of an arbitrary fiber, and satisfies the relation $\beta(xy) = \beta(x)\beta(y)$. One observes that there are

$$(7.7)\qquad \bar{v}_1 \in H^1(E, Z_p),\ \bar{v}_2 \in H^2(E, Z_p) \quad \text{such that} \quad \beta(\bar{v}_1) = v_1,\ \beta(\bar{v}_2) = v_2,$$

and hence that the homomorphism d_n in (7.6) is trivial since v_1 and v_2 generates the cohomology ring of $\Delta_{(np-n-1,p)}$ mod p. Thus, some usual arguments in spectral homology theory and, in particular, those to establish the sequence (7.6) together with (7.7) prove that E has the same cohomology ring mod p of the product $S_n^1 \times \Delta_{(np-n-1,p)}$.

We shall write also

$$(7.8)\qquad Q^s(E, Z_p) = \text{the kernel of the homomorphism } \beta \text{ in } (7.6).$$

The pair (C, E) has cohomology ring mod p with trivial multiplication (i.e., we have always $xy = 0$ for any cohomology classes x, y mod p of the pair (C, E)). In fact, since C, up to a homeomorphism, is the space $S_n^1 \cup E \times \langle 0, 1\rangle$ with the topology obtained by identifying, in $E \times \langle 0, 1\rangle$, the subset $E \times (0)$ to S_n^1 in a natural manner by means of the projection ϕ: $E \to S_n^1$ of $\mathcal{B}$, we notice that the multiplication of the cohomology ring of the pair $(E \times \langle 0, 1\rangle, E \times ((0) \cup (1))$ is trivial.

8. **A basic relation.** We shall write

$$M = \vartheta_{np} - (C - E)$$

and consider the following diagram (4.1) in which ι_0, ι_1 etc. are homomorphisms induced by inclusion $E \subset M$, $(M, E) \subset (\vartheta_{np}, C)$, etc., and δ, δ'

are coboundary homomorphisms.

$$(8.1)\qquad \begin{array}{ccccccc} & & H^{s+1}(C,\ E;\ J) & \overset{\iota_1'}{\longleftarrow} & H^{s+1}(\vartheta_{np},\ M;\ J) & & \\ & \overset{\delta'}{\nearrow} & & & & \overset{\iota_2'}{\searrow} & \\ H^s(E,\ J) & \longleftarrow & H^s(M,\ J) & & & & H^{s+1}(\vartheta_{np},\ J) \\ & \underset{\delta}{\overset{\iota_0}{\searrow}} & & & & \overset{\iota_2}{\nearrow} & \\ & & H^{s+1}(M,\ E;\ J) & \overset{\iota_1}{\longleftarrow} & H^{s+1}(\vartheta_{np},\ C;\ J) & & \end{array}$$

Clearly, ι_1 and ι_1' are onto-isomorphisms by excision, and the commutativity

$$(8.2)\qquad \iota_2\iota_1^{-1}\delta = \iota_2'\iota_1'^{-1}\delta'$$

holds by some usual arguments. Also, using the exactness of the cohomology sequences of the pair (C, E) and (ϑ_{np}, M), the assumption $n \geqq 3$, and (5.4), (7.2), we see that $H^1(M, Z_p)$, $H^2(M, Z_p)$ are cyclic groups of order p with generators

$$b_1 \in H^1(M, Z_p), \qquad b_2 \in H^2(M, Z_p)$$

such that

$$(8.3)\qquad \iota_0(b_1) = \bar{v}_1, \qquad \iota_0(b_2) = \bar{v}_2.$$

Finally, since S_n^1 is a deformation retract of C, we verify by (6.6) that

$$(8.4)\qquad \begin{aligned} \iota_2\iota_1^{-1}: H^s(M,\ E; Z_p) &\approx H^s(\vartheta_{np}, Z_p) \quad \text{for all } s \neq 0,\ n+1, \\ H^{n+1}(M,\ E; Z_p) &\approx Z_p. \end{aligned}$$

We shall write

$$(8.5)\qquad \begin{aligned} a_{n+1} = \mu\delta\bar{w}_n &\in \bar{w}H^{n+1}(M,\ E; Z_p) \qquad \text{with } \mu \in Z_p,\ \neq 0, \\ g_n &\in H^n(M,\ E; Z_p) \qquad \text{with } \iota_2\iota_1^{-1}(g_n) = g_n^{\#}. \end{aligned}$$

We assert that

$$(8.6)\qquad Q^{s+1}(\vartheta_{np}, Z_p) = \iota_2\iota_1^{-1}\delta Q^s(E, Z_p)$$

where $Q^{s+1}(\vartheta_{np}, Z_p)$ and $Q^s(E, Z_p)$ are given in (5.7), (7.8). In fact, it is easy to verify first that $Q^{s+1}(C,\ E;\ Z_p) = \delta' Q^s(E,\ Z_p)$ with $Q^s(C,\ E;\ Z_p)$ given in (7.3). In ϑ_{np}^0, we may take an open np-cell $W \subset \Delta_{np}^0 \cap (C_0 - E_0 - V_0)$ which T leaves invariant. Then, since the inclusions $(C_0,\ C_0 - W) \subset (C_0,\ E_0 \cup V_0)$ and $(\vartheta_{np}^0,\ \vartheta_{np}^0 - W) \subset (\vartheta_{np}^0,\ \Gamma_{np}^0)$ induce obviously onto-isomorphisms: $H^s(C_0,\ E_0 \cup V_0;\ Z_p) \approx H^s(C_0,\ C_0 - W;\ Z_p)$ and $H^s(\vartheta_{np}^0,\ \Gamma_{np}^0;\ Z_p) \approx H^s(\vartheta_{np}^0,\ \vartheta_{np}^0 - W;\ Z_p)$, it follows from (4.3) that the inclusions $(C,\ C - \bar{I}(W)) \subset (C,\ E \cup V)$ and $(\vartheta_{np},\ \vartheta_{np} - \bar{I}(W)) \subset (\vartheta_{np},\ \Gamma_{np})$ induce onto-isomorphism: $H^s(C,\ E \cup V;\ Z_p) \approx H^s(C,\ C - \bar{I}(W);\ Z_p)$ and $H^s(\vartheta_{np},\ \Gamma_{np};\ Z_p) \approx H^s(\vartheta_{np},\ \vartheta_{np} - \bar{I}(W);\ Z_p)$. This gives $\iota_2'\iota_1'^{-1}$: $Q^s(C,\ E;\ Z_p) \approx Q^s(\vartheta_{np},\ Z_p)$ by definition of these groups and some usual arguments. Then, using (8.2) we complete the proof of (8.6).

We shall write

$$
(8.7)\quad \begin{aligned} a^{\#}_{n+2k+1} &= \iota_2\iota_1^{-1}(a_{n+1}b_2^{k}) \in H^{n+2k+1}(\vartheta_{np}, Z_p) \quad \text{for } 0 \leqq 2k \leqq np-n-1,\\ a^{\#}_{n+2k+2} &= \iota_2\iota_1^{-1}(a_{n+1}b_1b_2^{k}) \in H^{n+2k+2}(\vartheta_{np}, Z_p) \quad \text{for } 0 \leqq 2k \leqq np-n-2. \end{aligned}
$$

Clearly, $\bar{w}_n=\phi^*(w_n)\in Q^s(E, Z_p)$, so $\bar{w}_n\bar{v}_2^k\in Q^{n+2k}(E, Z_p)$, $\bar{w}_n\bar{v}_1\bar{v}_2^k\in Q^{n+2k+1}(E, Z_p)$. Then, since $\delta(\bar{w}_n\bar{v}_2^k)=\mu a_{n+1}b_2^k$, $\delta(\bar{w}_n\bar{v}_1\bar{v}_2^k)=\mu a_{n+1}b_1b_2^k$, by (8.6) the $a_s^{\#}$ in (8.7) generates the cyclic group $Q^s(\vartheta_{np}, Z_p)$ and the $a_{n+1}b_2^k$, $a_{n+1}b_1b_2^k$ in (8.7) are not zero.

We shall establish the basic relation:

$$
(8.8)\qquad g_nb_2 = a_{n+1}b_1
$$

with a_{n+1} in (8.5) suitably chosen. In fact, since b_1, b_2 generates respectively $H^1(M, Z_p)$, $H^2(M, Z_p)$, using some usual arguments in cohomology theory, we conclude that there is

$$
b_2^* \in H^2(M, Z)
$$

of order p such that $b_2=b_2^*$ mod p and $\delta^*b_1=\lambda b_2^*$ with $\lambda\in Z$, $\neq 0$, where δ^* is the coboundary homomorphism of the coefficient sequence $0\to Z\to Z\to Z/pZ\to 0$. Also, since S_n^1 is a deformation retract of ϑ_{np}, by (5.4) and (6.6) we have $H^{n+2}(M, E; Z)=H^n(M, E; Z)=0$ and $H^{n+1}(M, E; Z)\approx Z_p$. Let $a_{n+1}^*\in H^{n+1}(M, E; Z)$ be a generator and let $b_{n+3}^*=a_{n+1}^*b_2^*$. Then $b_{n+3}^*\neq 0$ and $pb_{n+3}^*=0$. For, $a_{n+1}^*b_2^*$ mod p is $\nu a_{n+1}b_2$ with $\nu\in Z_p$, $\neq 0$, and $a_{n+1}b_2\neq 0$. That $pb_{n+3}^*=0$ is obvious.

Let us choose a covering of M with nerve N such that b_2^*, a_{n+1}^* are represented by cocycles $x_2\in C_2(N, Z)$, $x_{n+1}\in C_{n+1}(N, N_0; Z)$ with $px_2\sim 0$, $px_{n+1}\sim 0$ where N_0 denotes the subcomplex of N consisting of all simplexes whose nucleus meets E. Let $x_1\in C_1(N, Z)$, $x_n\in C_n(N, N_0; Z)$ with $\delta x_1=px_2$, $\delta x_n=px_{n+1}$. Then, x_1, x_2, x_n, x_{n+1} mod p represent respectively λ_1b_1, λ_2b_2, λ_3g_n, λ_4a_{n+1} with $\lambda_i\in Z_p$, $\neq 0$, and $x_{n+1}x_2$ represents ωb_{n+3}^* with $\omega\in Z$, $\neq 0$. Now $H^{n+2}(M, E; Z_p)\approx Z_p$ is generated by a_{n+2}; clearly $\delta^*a_{n+2}=\lambda' b_{n+3}^*$. Thus, since $\delta(x_{n+1}x_1)=(-)^npx_{n+1}x_2$, and $\delta(x_nx_2)=px_{n+1}x_2$, $g_nb_2=\sigma a_{n+1}b_1$, $\sigma\in Z_p$, $\neq 0$. Write σa_{n+1} simply as a_{n+1} so that we get (8.8).

Finally, let us remark here that a_{n+1} in (8.5) will be so chosen that (8.8) holds. It is obvious from (8.8) that

$$
(8.9)\qquad g_nb_2^{k+1} = a_{n+1}b_1b_2^{k}.
$$

9. **Cyclic reduced powers of** $Q^s(\vartheta_{np}, Z_p)$. This section contains only computations. Explicit relations between iterated powers are given for $Q^s(\vartheta_{np}, Z_p)$.

(I) $p=2$. It is easy to see from (5.4) that the Bockstein homomorphism maps $H^{2k}(S_n * S_n, Z_2)$ onto $H^{2k+1}(S_n * S_n, Z_2)$, $2\leqq 2k<n$. Thus, by (5.4) and (6.2),

$$(9.1)\qquad Sq^1 g_n^{\#} = 0, \qquad Sq^j g_n^{\#} = a_{n+j}^{\#}, \qquad 2 \leqq j \leqq n.$$

We verify $b_1^2 = b_2$, using (7.7), (8.3). As in [11], we have $Sq^j b_1^q = c(j, q; Z_2) b_1^{q+j}$ where $c(j, q; Z_2)$ is the mod 2 value of the binomial coefficient $C_{q,j}$ (with the convention: $C_{q,j} = 0$ if $j > q$). Write $a_{n+1+q} = a_{n+1} b_1^q$. Then, since $Sq^j a_{n+1} = \delta Sq^j \bar{w}_n = \delta\phi^* Sq^j w_n = 0$ for $j > 0$, the formula of Cartan gives $Sq^j a_{n+1+q} = c(j, q; Z_2) a_{n+1+q+j}$, $q + j + 1 \leqq n$. Thus, using (8.7),

$$(9.2)\qquad Sq^j a_{n+1+q}^{\#} = c(j, q; Z_2) a_{n+1+q+j}^{\#}, \qquad q + j + 1 \leqq n.$$

We have[9]

$$(9.3)\qquad Sq^j g_n^{\#} = Sq^{2^{i_1}} Sq^{2^{i_2}} \cdots Sq^{2^{i_{j'}}} g_n^{\#}, \qquad 2 \leqq j \leqq n,$$

for some integers $i_{j'} > i_{j'-1} > \cdots > i_1 \geqq 0$. In fact, if j is a power of 2, there is nothing to prove. If this is not the case, we may write $j = 2^k(2r+1)$ with $r > 0$, and hence, using (9.1), (9.2), and the argument in [11], $Sq^j g_n^{\#} = a_{n+j}^{\#} = Sq^{2^k} a_{n+j-2^k}^{\#} = Sq^{2^k} Sq^{j-2^k} g_n^{\#}$. An induction on j completes the proof.

It is easily verified by (9.1), (9.2) that the following (9.4) always holds:

$$(9.4)\qquad Sq^j Sq^{2^k - j} g_n^{\#} = 0 \qquad \text{for } 0 < j < 2^k.$$

(II) $p > 2$. If $n = 3$, $\mathcal{P}^k g_n \neq 0$ only when $k = 0$, 1 and $\mathcal{P}^1 g_n^{\#} = \lambda a_{n+2p-2}^{\#}$ with $\lambda \in Z_p$, $\neq 0$ by (6.2), (6.4). Hereafter, we assume $n \geqq 4$.

We verify first that $\mathcal{P}^k a_{n+1} \neq 0$ and $\mathcal{P}^k b_1 \neq 0$ only when $k = 0$ so that $\mathcal{P}^k a_{n+1} b_1 \neq 0$ only when $k = 0$ by the formula of Steenrod-Cartan [9]. We have $\mathcal{P}^k b_2^q = c(k, q; Z_p) b_2^{q+k(p-1)}$ where $2(q + k(p-1)) \leqq np - n - 1$ and $c(k, q; Z_p)$ is the mod p value of the binomial coefficient $C_{q,k}$. Also, $\mathcal{P}^k b_2 \neq 0$ only when $k = 0$, 1 and $\mathcal{P}^1 b_2 = b_2^p$.

Using the relation (8.8) and the formula of Steenrod-Cartan,

$$(9.5)\qquad (\mathcal{P}^k g_n) b_2 + (\mathcal{P}^{k-1} g_n) b_2^p = 0, \qquad k > 0.$$

Then, since $\mathcal{P}^k g_n^{\#} \in Q^{n+2k(p-1)}(\vartheta_{np}, Z_p)$, by (8.4) we conclude $\mathcal{P}^k g_n = -\mathcal{P}^{k-1} g_n b_2^{p-1}$, $k > 0$. It follows therefore from (8.9), (8.7), and (9.5) that

$$(9.6)\qquad \mathcal{P}^k g_n^{\#} = (-)^k a_{n+2k(p-1)}^{\#}, \qquad 2 \leqq 2k \leqq n.$$

With arguments parallel to that used in the case (I) before, we deduce that

$$(9.7)\qquad \begin{aligned} \mathcal{P}^k a_{n+2q+1}^{\#} &= c(k, q; Z_p) a_{n+2(q+k(p-1))+1}^{\#}, & 2(q + k(p-1)) &\leqq n(p-1) - 1. \\ \mathcal{P}^k a_{n+2q+2}^{\#} &= c(k, q; Z_p) a_{n+2(q+k(p-1))+2}^{\#}, & 2(q + k(p-1)) &\leqq n(p-1) - 2. \end{aligned}$$

A criterion for evaluating $c(k, q; Z_p)$ for $p = 2$ was given in [11]. A straight-

---

(9) See Adem [1]. Generalizations of results in [1] to cyclic reduced powers $\mathcal{P}^k$ mod p with p prime >2 had been also obtained by Adem.

forward generalization to any prime p is the following: Let

$$k = \sum_{i=0}^{m} a_i p^i, \qquad q = \sum_{i=0}^{m} b_i p^i \tag{9.8}$$

be the p-adic expansion of k, q. Then $c(k, q; Z_p) = c(a_1, b_1; Z_p)\, c(a_2, b_2; Z_p) \cdots c(a_m, b_m; Z_p)$. Thus,

(9.9) *$c(k, q; Z_p) \neq 0$ if and only if, in the p-adic expansions* (5.8) *of k, q, $a_i \leqq b_i$, $i = 0, 1, \cdots, m$.*

(9.10) *$\mathcal{P}^k g_n^{\#}$, $2 \leqq 2k \leqq n$, can be expressed as*

$$\mathcal{P}^k g_n^{\#} = \mu \mathcal{P}^{p^{i_1}} \mathcal{P}^{p^{i_2}} \cdots \mathcal{P}^{p^{i_{k'}}} g_n^{\#}(^9), \qquad \mu \in Z_p,\ 2 \leqq 2k \leqq n, \tag{9.11}$$

for some integers $i_{k'} \geqq i_{k'-1} \geqq \cdots \geqq i_1 \geqq 0$ where the number of i_j's which are equal to a given i' is just the coefficients $a_{i'}$ in the p-adic expansion (9.8) *of k.*

In fact, if k is a power of p, there is nothing to prove. Suppose that this is not the case. Let i_1 be the least i such that $a_i \neq 0$ with k written as in (5.8), and let $r = \sum_{i=i_0+1}^{m} a_i p^{i-(i_1+1)}$. Then, either (i) $a_{i_1} > 1$ or (ii), $a_{i_1} = 1$, $r \geqq 1$. Write $t = k(p-1) - 1$. In case (i), a simple calculation gives $t - p^{i_1}(p-1) = p^{i_1+1}(r + a_{i_1} - 2) + (p - a_{i_1})p^{i_1} + \sum_{i=0}^{i_1-1} (p-1)p^i$ with $r + a_{i_1} - 2 \geqq 0$, $p > p - a_{i_1} > 0$. It follows therefore from (9.9) that $c(p^{i_1}, t - p^{i_0}(p-1); Z_p) \neq 0$, and hence from (9.5), (9.6) that

$$\mathcal{P}^k g_n^{\#} = \mu' \mathcal{P}^{p^{i_1}} \mathcal{P}^{k - p^{i_1}} g_n^{\#}, \qquad \mu' \in Z_p. \tag{9.12}$$

In case (ii), a simple calculation gives $t - p^{i_1}(p-1) = p^{i_0+1}(r(p-1)-1) + \sum_{i=0}^{i_0} (p-1)p^i$, and hence, using similar arguments as before, we have (9.12). An induction on k completes the proof of our statement.

(9.13) *The expression*

$$\mathcal{P}^k g_n^{\#} = \mu \mathcal{P}^{p^{i_1}} \mathcal{P}^{p^{i_2}} \cdots \mathcal{P}^{p^{i_{k'}}} g_n^{\#}, \qquad \mu \in Z_p,\ 2 \leqq 2k \leqq n,$$

for some $i_{k'} \geqq i_{k'-1} \geqq \cdots \geqq i_1 \geqq 0$ is unique.

We need prove that any expression

$$\mathcal{P}^k g_n^{\#} = \mu' \mathcal{P}^{p^{i'_1}} \mathcal{P}^{p^{i'_2}} \cdots \mathcal{P}^{p^{i'_{k''}}} g_n^{\#}, \quad \mu' \in Z_p,\ 2 \leqq 2k \leqq n, \tag{9.14}$$

for some $i'_{k''} \geqq i'_{k''-1} \geqq \cdots \geqq i'_1 \geqq 0$ is the same as that in (9.10). It is easy to see that, in (9.11), (9.14), $k = \sum_{j=1}^{k'} p^{i_j} = \sum_{j=1}^{k''} p^{i'_j}$, and we assert $i'_1 = i_1$. In fact, if this is not the case, e.g., $i'_1 < i_1$, we shall have $i'_1 = i'_2 = \cdots = i'_p$, and a simple calculation, using (9.5), (9.6) will give that, in (9.14), $\mathcal{P}^k g_n^{\#} = 0$, contradicting a well-established result. Therefore $i'_1 = i_1$, and hence

$$\mu \mathcal{P}^{p^{i_2}} \cdots \mathcal{P}^{p^{i_{k'}}} g_n^{\#} - \mu' \mathcal{P}^{p^{i'_2}} \cdots \mathcal{P}^{p^{i'_{k''}}} g_n^{\#} = 0.$$

Thus, an induction on k will complete the proof.

As in (9.4), we verify by (9.6), (9.7) that

$$\mathcal{P}^k\mathcal{P}^{p^i-k}g_n^{\#} = 0 \qquad \text{for } 0 < k < p^i. \tag{9.15}$$

III. Homotopy groups and 2-fold and 3-fold cyclic products of spheres

10. **The problem.** Let K be a connected complex. Consider the q-fold symmetric product K^{Σ_q} of K. We regularly imbed K in K^{Σ_q} in the sense of §3 with reference point $s_0 \in K$, and denote by $\iota: \pi_s(K) \to \pi_s(K^{\Sigma_q})$ the usual injection homomorphism between homotopy groups. Let $\alpha \in \pi_s(K)$. If $\iota(\alpha) = 0$, we shall say that α is killed by q-fold symmetrization of K. We assert that the behavior of α being killed by symmetrization is independent of the choice of the reference point $s_0 \in K$ to define the regular imbedding. In fact, given any other $s_0' \in K$, there is a homotopy equivalence $f: K \to K$ with $f(s_0) = s_0'$, and, as was shown in §1, f^{Σ_q} is a homotopy equivalence: $K^{\Sigma_q} \to K^{\Sigma_q}$. Also, after imbedding K into K^{Σ_q} with respect to both the reference points s_0, $s_0' \in K$, we see that the partial map $f^{\Sigma_q} | s_0(K)$ is the same as f. Homotopy equivalence induces isomorphisms between homotopy groups. Our assertion is thus clear.

We may give also the meaning for the expression that the elements of $\pi_s(K)$ are killed in Γ-products of K in the same way as above with Σ_q replaced by Γ. Let $\Gamma \subset \Sigma_q$. By the natural map $\eta_{\Gamma\Sigma_q}: K^\Gamma \to K^{\Sigma_q}$ (§3), it is clear that $\alpha \in \pi_s(K)$ is killed in K^{Σ_q} if it is killed in K^Γ.

Let $s_0 \in K$ be given. We imbed K^{Σ_q} in $K^{\Sigma_{q+1}}$ as follows. Let $f_q: K^q \to K^{q+1}$ be the map sending $(x_1, x_2, \cdots, x_q)$ to $(x_1, x_2, \cdots, x_q, s_0)$. It is clearly an into-homeomorphism. Form K^{Σ_q} and $K^{\Sigma_{q+1}}$ respectively over K^q and K^{q+1}. f_q gives rise naturally to a map $f_q': K^{\Sigma_q} \to K^{\Sigma_{q+1}}$. It is easily seen that f_q' is an into-homeomorphism. Let us identify K^{Σ_q} and $f_q'(K^{\Sigma_q})$ under f_q'. We have thus the imbedding: $K = K^{\Sigma_1} \subset K^{\Sigma_2} \subset K^{\Sigma_3} \subset \cdots$. Clearly, $\alpha \in \pi_s(K)$ is killed by $(q+1)$-fold symmetrization, if it is killed by q-fold symmetrization.

We consider the simple case $K = S_n$. We have the problem:

How many homotopy groups of S_n could be killed by symmetrization?

The results in later sections give that 2-fold symmetrization of S_n kills $\pi_{n+1}(S_n)$, $\pi_{n+2}(S_n)$, the 2-primary subgroup of $\pi_{n+3}(S_n)$ for $n \geqq 5$; 3-fold symmetrization kills the 3-primary subgroup of $\pi_{n+3}(S_n)$ for $n \geqq 3$. Symmetrization kills all homotopy groups $\pi_s(S_2)$ for $s > 2$, since the q-fold symmetric product of S_2 is homeomorphic to the complex projective space M_{2q} of (topological) dimension $2q$. (Take S_2 as the extended complex plane and express M_{2q} by homogeneous coordinates in complex numbers. Define $\phi: S_2^q \to M_{2q}$ by taking $\phi(z_1, z_2, \cdots, z_q) = (a_0, a_1, \cdots, a_q)$ if the z_i's are the n roots of the equation $\sum_{i=0}^q a_i x^i = 0$. Identifying the inverse image under ϕ of every point of M_{2q} gives a homeomorphism between S^{Σ_q} and M_{2q}.) It seems true that more and more homotopy groups of spheres are killed by symmetrization. Perhaps one of the reasons might be illustrated by the proof of (13.7), in which we compare the cohomology groups of a symmetric product and those of an Eilenberg-MacLane complex.

Very little is known concerning the homology properties of symmetric products([2]). It follows easily from (3.5) and (6.2) that, if $g_n \in H^n(S_n, Z)$ is the generator and μ: $H^n(S_n, Z) \to H^n(S_n^{\Sigma_q}, Z)$ is the into-isomorphism (3.2), then, in $S_n^{\Sigma_q}$, we have $\mathcal{P}^k\mu(g_n) \neq 0$ for all cyclic reduced powers mod p and p prime $\leqq q$ and $2k \leqq n$.

11. **The cellular decomposition of $S_n * S_n$ of Steenrod.** As the integral cohomology groups of $\vartheta_{np}(n \geqq 2)$ are computed as in (5.4), its integral homology groups can be easily obtained by duality theorems. ϑ_{np} is simply connected by (1.4). Thus, following a statement given in later Appendix, there is a finite cellular complex([3]) ϑ'_{np} of the same homotopy type of ϑ_{np}, composed of oriented open cells C's, D's as follows:

$$\vartheta'_{np} = (C_0 \cup C_n \cup C_{n+2} \cup C_{n+3} \cup \cdots \cup C_{np}) \tag{11.1}$$
$$\bigcup_{r=2}^{p-1} (\overset{1}{D}_{nr} \cup \overset{2}{D}_{nr} \cup \cdots \cup \overset{C_{p,r}/p}{D_{nr}}), \qquad (n \geqq 2),$$

where (α) the subscript of C, D denotes the dimension of the cell, (β) the homotopy boundary([10]) of C_{n+2k} is contained in the $(n+2(k-1))$th skeleton of ϑ'_{np}, (γ) the homotopy boundary of $\overset{i}{D}_{nr}$ is contained in the $(nr-2)$th skeleton of ϑ'_{np}, and (δ) the homology boundary([11]) of C_{n+2k+1} is pC_{n+2k}.

We do not know apparently the homotopy boundaries of the open cells in (11.1). For $p=2$, an explicit cellular decomposition of ϑ_{np} was given by Steenrod([12]). The basic lemma for this is the following (11.2).

(11.2) *$S_n * S_n$ is homeomorphic to the space obtained by first suspending $S_{n-1} * S_{n-1}$ and then adjoining a $2n$-cell to the suspension.*

Let S_n be the set in the Euclidean $(n+1)$-space defined by the equation $\sum_{i=1}^{n+1} u_i = 1$. S_n is cut by the hyperplane $u_{n+1}=0$ into upper cap E_+ and lower cap E_- and $S_{n-1}=E_+ \cap E_-$. We shall write $[x, y] = \bar{l}(x, y) \in S_n * S_n$ for $(x, y) \in S_n \times S_n$. Let K be the subset of points $[x, y]$ in $S_n * S_n$ such that x and y have the same latitude in S_n relative to S_{n-1}. Clearly K is homeomorphic to the suspension of $S_{n-1} * S_{n-1}$. For any point x of S, let $\theta(x)$ be its latitude ($+$ in E_+, $-$ in E_-), $-\pi/2 \leqq \theta(x) \leqq \pi/2$. Define ϕ: $E_+ \times E_- \to S_n * S_n$ as follows. Let $x \in E_+$, $y \in E_-$. If $\theta(x) \geqq -\theta(y)$, let y' be the point on the longitudinal circle of y with latitude $\theta(y)+(\theta(x)+\theta(y)) = \theta(x)+2\theta(y)$ and set $\phi(x, y) = [x, y']$. If $\theta(x) \leqq -\theta(y)$, let x' be the point on the longitudinal circle

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([10]) See [14, p. 221]. We call the partial map over the boundary $\vartheta\sigma_n$ of the characteristic map f: $\sigma_n \to \bar{e}_n$ the homotopy boundary of the open cell e_n.

([11]) See later Appendix.

([12]) The remainder of this section are unpublished results of N. E. Steenrod. Thanks are due to him for his permission to include these results here.

of x with latitude $\theta(x)+(\theta(x)+\theta(y))=2\theta(x)+\theta(y)$ and set $\phi(x, y)=[x', y]$. Note that $\theta(x)=-\theta(y)$ implies $\phi(x, y)=[x, y]$ in both cases, so ϕ is a (continuous) map.

We show that ϕ is onto. Let $[x, y]$ be given. If $\theta(x)=\theta(y)$ and $x, y\in E_+$, let y_1 be the point on the equator of S_n having the longitude of y. Then $\phi(x, y_1)=[x, y]$; for $\theta(y_1)=0$ implies first case, so $\phi(x, y_1)=(x, y_1')$ where $\theta(y')=\theta(x)+2\theta(y)=\theta(x)$, hence $y_1'=y$. Similarly, if $\theta(x)=\theta(y)$ and $x, y\in E_-$, we work on x.

Since $[x, y]=[y, x]$ in $S_n * S_n$, we need consider only the case $\theta(x)>\theta(y)$. We must consider various cases.

(i) $$\theta(x)+\theta(y)\leqq 0$$

which implies $\theta(y)\leqq 0$. Solve equation $\theta(x)=2\theta(x)+\theta(y)$,

$$\theta(x_1) = [\theta(x)-\theta(y)]/2,$$

and let x_1 have the same longitude as x. The $\theta(x_1)\geqq 0$ since $\theta(x)\geqq\theta(y)$ and $\theta(x_1)+\theta(y)=[\theta(x)+\theta(y)]/2\leqq 0$, so Case 2 of definition applies and $\phi(x_1, y)=[x, y]$.

(ii) $$\theta(x)+\theta(y)\geqq 0.$$

Solve equation $\theta(y)=\theta(x)+2\theta(y_1)$,

$$\theta(y_1) = [\theta(y)-\theta(x)]/2 \leqq 0,$$

and let y_1 have the same longitude as y. Then Case 1 of definition applies for $\theta(y_1)+\theta(x)=(\theta(y)+\theta(x))/2\geqq 0$, so $\phi(x, y_1)=(x, y)$. Thus, the map ϕ is onto.

Now suppose

$$\phi(x, y) = \phi(x_1, y_1), \qquad x, x_1\in E_+, y, y_1\in E_-.$$

Suppose first that $\theta(x)+\theta(y)\geqq 0$. Then $\phi(x, y)=[x, y']$ as in Case 1 of definition, and we have $\theta(x)+\theta(y')=2[\theta(x)+\theta(y)]\geqq 0$. Now $\theta(x_1)+\theta(y_1)<0$ would lead to $\phi(x_1, y_1)=(x_1', y_1)$ with $\theta(x_1')+\theta(y_1)=2[\theta(x_1)+\theta(y_1)]<0$ which contradicts $\phi(x, y)=\phi(x_1, y_1)$. So we must have $\theta(x_1)+\theta(y_1)\geqq 0$. Hence $\phi(x_1, y_1)=(x_1, y_1')$ with $\theta(x_1)+\theta(y_1')=2[\theta(x_1)+\theta(y_1)]$. Suppose first that $x_1=x$, $y_1'=y'$. Then

$$\theta(y') = \theta(x)+2\theta(y), \qquad \theta(y_1') = \theta(x_1)+2\theta(y_1)$$

imply $\theta(y)=\theta(y_1)$, so we obtain $(x, y)=(x_1, y_1)$, or else $x=x_1=$north pole of S_n and $y, y_1=$any two points on S_{n-1}. Suppose then that $x_1=y'$ and $x=y_1'$. By equations above we get $\theta(y)=-\theta(y_1)$. Since both $y, y_1\in E_-$ we must have $\theta(y)=\theta(y_1)=0$. This implies that (x, y) and (x_1, y_1) are both in $E_+\times S_{n-1}$. Assuming $\theta(x)+\theta(y)<0$ we deduce similarly that $\phi(x, y)=\phi(x_1, y_1)$ implies $(x, y)=(x_1, y_1)$ or that both points are in $S_{n-1}\times E_-$.

To sum up, ϕ: $E_+\times E_-\to S_n * S_n$ maps the boundary $E_+\times S_{n-1}\cup S_{n-1}\times E_-$

of $E_+ \times E_-$ onto $K \subset S_n * S_n$ and maps the interior of $E_+ \times E_-$ onto $S_n * S_n - K$ topologically. (11.2) is proved.

Obviously, using (11.2), a cellular decomposition

$$S_n * S_n = C_0 \cup C_n \cup C_{n+2} \cup \cdots \cup C_{2n} \qquad (n \geqq 2) \tag{11.3}$$

can be constructed by induction on n where the subscript of C denotes the dimension number of the cell. We start with $n=2$. $S_2 * S_2$ is the complex projective plane, and we take the desired cellular decomposition of $S_2 * S_2$ as that given in [10, p. 310]. Suppose that (11.3) has been already constructed for $S_{n-1} * S_{n-1}$. We decompose $K \subset S_n * S_n$ first into $K = C_0 \cup C_n \cup C_{n+2} \cup \cdots \cup C_{2n-1}$ as a homeomorphic image of the suspension of the decomposition of $S_{n-1} * S_{n-1}$ (by the homeomorphism given in above proof) and then adjoin a $2n$-cell to K by means of the map ϕ. Clearly, the partial map $\phi | S_{n-1} * S_{n-1}$ coincides with the identification map $\bar{I}: S_{n-1} \times S_{n-1} \to S_{n-1} * S_{n-1}$, and the involution T acting on $S_{n-1} \times S_{n-1}$ preserves or reverses orientation according as n is odd or even. Let the C_i's in (11.3) be oriented arbitrarily. By a consideration of the partial map $\phi | (E_+ \times S_{n-1} \cup S_{n-1} \times E_-)$, we find that, in (11.3), the homology boundary of C_{2n} is $\pm 2C_{2n-1}$ or 0 according as n is odd or even. Thus, by the inductive construction of (11.3) and by some usual property of suspension, we conclude that the homology boundary of C_{n+j} $(j \geqq 3)$ is $\pm 2C_{n+j-1}$ or 0 according as j is odd or even. (11.3) can be also applied to show that $Sq^j g_n^\# \neq 0$, $2 \leqq j \leqq n$. In fact, using (6.1) and the commutativity between Sq^i and suspension, we have first $Sq^{2k} g_n^\# \neq 0$, $2 \leqq 2k \leqq n$. Then, since the Bockstein homomorphism maps $H^{2k}(S_n * S_n, Z_2)$ onto $H^{2k+1}(S_n * S_n, Z_2)$, we have $Sq^{2k+1} g_n^\# \neq 0$, $2 \leqq 2k+1 \leqq n$.

12. **p-primary subgroups of $\pi_q(S_n)$, $n \geqq 3$.** It is known that the p-primary subgroup $C(\pi_q(S_n), p)$ of $\pi_q(S_n)$, $n \geqq 3$, vanishes for $n < q < n+2p-3$, and $C(\pi_{n+2p-3}(S_n), p) \approx Z_p$ [5]. Let us give the characteristic property of $C(\pi_{n+2p-3}(S_n), p)$ by reduced powers.

Denote by $K(S_n)$ an Eilenberg-MacLane complex, which is cellular, and whose only nonvanishing homotopy group is $\pi_n(K(S_n))$ and is isomorphic to $\pi_n(S_n)$. If K is arbitrary cellular complex whose first nonvanishing homotopy group is $\pi_n(K) \approx \pi_n(S_n)$, using some standard method we may imbed K into a complex $K(S_n)$ such that the injection homomorphism: $\pi_n(K) \to \pi_n(K(S_n))$ is an onto-isomorphism. We say then that $K(S_n)$ is an Eilenberg-MacLane complex constructed starting with K. Any two complex $K(S_n)$ are of the same homotopy type. Clearly, $H^n(K(S_n), Z_p) \approx Z_p$. Let $\alpha_n \in H^n(K(S_n), Z_p)$ be a generator.

(12.1) *In $K(S_n)$, we have $\mathcal{P}^k \alpha_n \neq 0$, $0 \leqq 2k \leqq n$.*

In fact, it follows from (1.4) and (5.4) that $\pi_s(\vartheta_{np})$ vanishes for $1 \leqq s < n$ and $\approx Z$ for $s = n$. Construct $K(S_n)$, starting with ϑ_{np}. We may assume that $i^*(\alpha_n) = g_n^\#$ where i denotes the inclusion $\vartheta_{np} \subset K(S_n)$. Then (12.1) follows from (6.2) and the commutativity $\mathcal{P}^k i^* = i^* \mathcal{P}^k$.

Similarly, we may show that, in $K(S_n)$, $Sq^j\alpha_n \neq 0$, $2 \leqq j \leqq n$.

We shall write $K(S_n, p)$ for the cellular complex which is the union of the cellular complexes $S_n = K_1, K_2, K_3, \cdots$, where K_s is constructed by adjoining cells to K_{s-1}, by means of maps f_j of a q_s-sphere into K_{s-1} which represent a basis of the p-primary subgroup $C(\pi_q(K_{s-1}), p)$ of $\pi_{q_s}(K_{s-1})$ with $C(\pi_{q_s}(K_{s-1}), p)$ nonvanishing and q_s least. It is easy to verify by our construction that the p-primary subgroup of $\pi_q(K(S_n, p))$ vanishes for every $q > n$ and $\pi_n(K(S_n, p)) \approx \pi_n(S_n)$.

Write

$$E_m = \text{an oriented } m\text{-cell}(^8).$$

Denote by E_f the cellular complex constructed by adjoining E_{q+1} to S_n, by means of a map $f: \partial E_{q+1} \to S_n$, $q > n$. Clearly, $H^k(E_f, Z_p) \approx Z_p$ for $k = n$, $q+1$ and vanishes for other s. Let S_n be oriented and let $\beta_n \in H^n(E_f, Z_p)$ be the generator represented by S_n, and write

$$e_f \in \pi_q(S_n) \text{ for the element represented by } f.$$

Let $q = n + 2k(p-1) - 1$. We observe that $\mathcal{P}^k\beta_n \in H^{n+2k(p-1)}(E_f, Z_p)$ gives rise to an element $e_f' \in H^{n+2k(p-1)}(E_{n+2k(p-1)}, \partial E_{n+2k(p-1)}; Z_p)$ by identifying these two groups in a natural manner. For $n \geqq 3$, let us define

$$(12.2) \qquad \psi: \pi_{n+2k(p-1)-1}(S_n) \to H^{n+2k(p-1)}(E_{n+2k(p-1)}, \partial E_{n+2k(p-1)}; Z_p)$$

by taking $\psi(e_f) = e_f'$. ψ is then a homomorphism([13]).

(12.3) *The homomorphism ψ with $k=1$ maps $C(\pi_{n+2p-3}(S_n), p)$ isomorphically onto $H^{n+2(p-1)}(E_{n+2(p-1)}, \partial E_{n+2(p-1)}; Z_p)$ $(n \geqq 3)$. If $e \in \pi_{n+2p-3}(S_n)$ is of finite order prime to p, then $\psi(e) = 0$.*

Consider first $n = 3$. Since $C(\pi_q(S_3), p) = 0$, $3 < q < 2p$, in the complex $K(S_3, p)$ there is no cell of dimension greater than 3 and less than $2p+1$, and since $C(\pi_{2p}(S_3), p) \approx Z_p$ there is exactly one $(2p+1)$-cell with homotopy boundary $f: \partial E_{2p+1} \to S_3$ representing a nonzero element of $C(\pi_{2p}(S_3), p)$, and $K(S_3, p)$ contains thus E_f. Construct the cellular complex $K(S_3)$, starting with $K(S_3, p)$. Then, by the exactness of the homotopy sequence of the pair $(K(S_3), K(S_3, p))$, we have that the p-primary subgroup of $\pi_q(K(S_3), K(S_3, p))$ vanishes for $1 < q \leqq 2p+1$. Also, $\pi_q(K(S_3), K(S_3, p))$ has no element of order infinity for $1 < q \leqq 2p+1$. It follows therefore from the generalized Hurewicz theorem as given by J. Moore [5] that $H^{2p+1}(K(S_3), K(S_3, p); Z_p) = 0$. Then, using the exactness of the cohomology sequence of the pair $(K(S_3), K(S_3, p))$, (12.1) gives that, in E_f, $\mathcal{P}^1\beta_n \neq 0$. Hence, ψ is onto for $n = 3$. Using suspension, we see that ψ maps $C(\pi_{n+2p-3}(S_n), p)$ onto $H^{n+2(p-1)}(E_{n+2(p-1)}, \partial E_{n+2(p-1)}; Z_p)$. It is obvious that ψ maps an element of $\pi_{n+2p-3}(S_n)$ of finite order prime to p into 0.

([13]) It follows from some theorem of Adem that θ is 0 if k is not a power of p. For the case $p=2$, see [1].

13. On homotopy groups of 2-fold and 3-fold cyclic products of spheres. We regularly imbed S_n in ϑ_{np} and write

$$S_n \subset \vartheta_{np}.$$

Let S_n be oriented and let g_n^* be so chosen that the projection homomorphism: $H^n(\vartheta_{np}, Z) \to H^n(S_n, Z)$ maps g_n^* into the generator of $H^n(S_n, Z)$ represented by S_n. Write

(13.1) ξ = the injection homomorphism: $H^s(\vartheta_{np}, S_n; G) \to H^s(\vartheta_{np}, G)$,

where G is an arbitrary coefficient group. ξ is an isomorphism for $s > n$. As in (12.2), we define

(13.2) $$\psi: \pi_3(S_2) \to H^4(E_4, \partial E_4; Z)$$

by taking now $\psi(e_f) = e_f'$ with e_f' corresponding to $\beta_2^2 \in H^4(E_f, Z)$. ψ is an onto-isomorphism. For if $f = f_0$: $\partial E_4 \to S_2$ is a map with Hopf invariant 1, we have $\psi(e_{f_0}) \in H^4(E_4, \partial E_4; Z)$ represented by the oriented cell E_4 [10, p. 311]. Some usual arguments give then $\psi(te_{f_0}) = t\psi(e_{f_0})$ for any integer t.

(13.3) *The $(n+1)$th homotopy group of $S_n * S_n$ vanishes $(n \geqq 2)$. Thus, every map f: $\partial E_{n+2} \to S_n$ extends in $S_n * S_n$ over E_{n+2}.* Moreover, if F is such an extension of f, we have:

(13.4)
$$\psi(e_f) = F^*\xi^{-1}(g_2^{*2}) \in H^4(E_4, \partial E_4; Z), \qquad n = 2,$$
$$\psi(e_f) = F^*\xi^{-1}Sq^2 g_n^* \in H^4(E_{n+2}, \partial E_{n+2}; Z_2), \qquad n \geqq 3,$$

where ψ is the isomorphism (13.2) *or* (12.2) *with $k=1$, $p=2$, according as $n=2$ or $n \geqq 3$*(1).

In fact, $S_2 * S_2$ is the complex projective plane, so $\pi_3(S_2 * S_2) = 0$. For $n \geqq 3$, we observe that the subcomplex $C_0 \cup C_n \cup C_{n+2}$ of $S_n * S_n$ with respect to the cellular decomposition (11.3) is homeomorphic to the complex M_{n+2} considered by Steenrod [10, p. 311], and by arguments there, we verify that the boundary homomorphism

(13.5) $$\partial: \pi_{n+2}(S_n * S_n; W_n) \to \pi_{n+1}(W_n)$$

is onto, where $W_n = C_0 \cup C_n$ is an n-sphere. Clearly, $H_s(S_n * S_n, W_n; Z) = 0$ for $s < n+2$ and hence, by (1.4), the Hurewicz isomorphism gives $\pi_{n+1}(S_n * S_n; W_n) = 0$. So, by the exactness of the homotopy sequence of the pair $(S_n * S_n, W_n)$, we have $\pi_{n+1}(S_n * S_n) = 0$.

To prove (13.4), we need merely identify ∂E_{n+2} with S_n by the map f and use the definition of ψ. Then, F gives rise naturally to a map F': $E_f \to S_n * S_n$. Since the homomorphism induced by a map preserves cup product and square operations, we deduce thus (13.4) easily.

(13.6) *The $(n+2)$th homotopy groups of $S_n * S_n$ vanishes* $(n \geqq 2)$(1).

This can be proved by some arguments similar to those given before. For

$n=2$, this follows from the fact that the fourth homotopy group of the complex projective plane vanishes. Hence, every map: $S_4 \to W_2$ can be deformed in M_4 into a point. Suppose inductively that a map f: $S_{n+2} \to W_n$ for a certain $n \geqq 2$ can be deformed in M_{n+2} into a point s. Note that the suspension of the pair (M_{n+2}, W_n) is the same as the pair (M_{n+3}, W_{n+1}) (up to a homeomorphism) and the suspension of $\pi_{n+2}(W_n)$ is $\pi_{n+3}(W_{n+1})$ (for W_n and W_{n+1} are spheres). It follows that the suspension of f in M_{n+3} can be deformed into a broken line in M_{n+3}, and hence every map: $S_{n+3} \to W_{n+1}$ can be deformed into a point. This gives that the injection homomorphism: $\pi_{n+2}(W_n) \to \pi_{n+2}(S_n * S_n)$ is trivial.

Let $n \geqq 3$. As in the cellular decomposition (11.3), the homology boundary of C_{n+3} is $\pm 2C_{n+2}$, so $H_{n+2}(S_n * S_n, W_n; Z) \approx Z_2$ and as before we deduce $\pi_{n+2}(S_n * S_n, W_n) \approx Z_2$. This gives that the boundary homomorphism (13.5) is an onto-isomorphism. We get thus (13.6) easily.

In (12.2), let $n \geqq 4$, $k=2$, $p=2$. It was known that the 2-primary subgroup of $\pi_{n+3}(S_n) \approx Z_8$ for $n \geqq 5$ [4]. If f is a map: $S_7 \to S_4$ with Hopf invariant 1, $Sq^4\beta_4 =$ the generator of $H^8(E_f, Z_2)$ [1]. So ψ is onto for $n=4$. The suspension gives that ψ is onto for $n \geqq 5$, and hence $\pi_{n+3}(S_n)/2\pi_{n+3}(S_n)$ is characterized by Sq^4 for $n \geqq 5$.

(13.7) *The 2-primary subgroup $C(\pi_{n+3}(S_n * S_n), 2)$ of $\pi_{n+3}(S_n * S_n)$ vanishes for $n \geqq 5$. Thus every map $f:\partial E_{n+4} \to S_n$ which represents an element of $C(\pi_{n+3}(S_n * S_n), 2)$, $n \geqq 5$, extends in $S_n * S_n$ over E_{n+4}. Moreover, if F is such an extension, we have*:

$$(13.8) \qquad \psi(e_f) = F^*\xi^{-1}(Sq^4 g_n^*) \in H^{n+4}(E_{n+4}, \partial E_{n+4}; Z_2),$$

where ψ is the homomorphism (12.2) *with $k=2$, $p=2$.*

To prove this, we construct the Eilenberg-MacLane complex $K(S_n)$, starting with $S_n * S_n$. Consider the homomorphism i_* between homology groups and also between homotopy groups induced by the inclusion $S_n * S_n \subset K(S_n)$ and the dual homomorphism i^* between cohomology groups. Let $n \geqq 5$. We have that $H_s(S_n * S_n, Z_2) = H_s(K(S_n), Z_2) = 0$ for $1 \leqq s \leqq n-1$ and $s=n+1$, $H_s(S_n * S_n, Z_2) \approx Z_2 \approx H_s(K(S_n), Z_2)$ for $n+2 \leqq s \leqq n+4$ [2, pp. 661–662][14]. Obviously, $i_*: H_n(S_n * S_n, Z) \approx H_n(K(S_n), Z)$. Then, since, in $S_n * S_n$, $Sq^j g_n^* = 0$ for $2 \leqq j \leqq n$, we see easily by the commutativity $i^*Sq^i = Sq^i i^*$ that $i_*: H_s(S_n * S_n, Z_2) \to H_s(K(S_n), Z_2)$ is an into-isomorphism. Thus, $H_s(K(S_n), S_n * S_n; Z_2) = 0$ for $1 \leqq s \leqq n+4$ and by the generalized relative Hurewicz theorem [5], $\pi_s(K(S_n), S_n * S_n) \otimes Z_2 = 0$ for $1 \leqq s \leqq n+4$. This leads to that i_*:

$$C(\pi_s(S_n * S_n), 2) \approx C(\pi_s(K(S_n), 2) = 0 \qquad \text{for } n < s \leqq n+3.$$

Now, let f be a map: $\partial E_{n+4} \to S_n$, representing the generator of

[14] In [2, p. 662] only integral homology groups $A_i(Z)$ are given explicitly. However, we may then determine $A^i(\pi_n(S_n), Z_2)$ from $A_i(Z)$ by usual arguments in universal coefficient theorems and duality theorems.

$C(\pi_{n+3}(S_n), 2)$, $n \geqq 5$. f extends over E_{n+4} in $S_n * S_n$. If F is such an extension, then, since ψ is an onto-homomorphism (as was shown previously), we verify easily (13.8). The proof can be then completed by usual arguments.

(13.9) *Every map* $f: \partial E_{n+4} \to S_n$ *extends in* ϑ_{n3} *over* E_{n+4} *if it represents an element of* $C(\pi_{n+3}, (S_n), 3)$, $n \geqq 3$. *If* F *is such an extension, we have*

$$\psi(e_f) = F^* \xi^{-1} \mathfrak{P}^1 g_n^* \in H^{n+4}(E_{n+4}, \partial E_{n+4}; Z_3) \tag{13.10}$$

where ψ *is the homomorphism* (12.2) *with* $k=1$, $p=3$, *and the cyclic reduced power* $\mathfrak{P}^1$ *is defined for* $p=3$.

To prove (13.9), we consider the complex ϑ'_{n3} and its cellular decomposition in (11.1). C_n represents the generator g_n' of the infinite cyclic group $H^n(\vartheta'_{n3}, Z)$; C_{n+2} and C_{n+4} represent respectively elements of $H_{n+2}(\vartheta'_{n3}, Z)$, $H_{n+4}(\vartheta'_{n3}, Z)$ of order 3.

Write $S_n' = C_0 \cup C_n$, which is an n-sphere. Let $h_{n+1}: \partial E_{n+2} \to S_n'$ denote the homotopy boundary of C_{n+2}(10). We see easily that $\pi_{n+2}(\vartheta'_{n3}, S_n') \approx Z_3$. But the 3-primary subgroup of $\pi_{n+1}(S_n')$ vanishes; so $h_{n+1} \simeq 0$ in S_n'. Therefore, by usual arguments in homotopy theory, we shall assume that the complex ϑ'_{n3} is such that the homotopy boundary of C_{n+2} in S_n' is null; in other words, $C_0 \cup C_n \cup C_{n+2}$ is the union $S_n' \vee S'_{n+2}$ of an n-sphere and an $(n+2)$-sphere with one point in common.

Let $h_{n+3}: \partial E_{n+4} \to S_n' \vee S'_{n+2}$ denote the homotopy boundary of C_{n+4}. Let $e_{n+3} \in \pi_{n+3}(K_{n+3})$ be represented by h_{n+3} where K_{n+3} denotes the $(n+3)$-skeleton of ϑ'_{n3}. Then, since C_{n+4} represents the element of $H_{n+4}(\vartheta'_{n3}, Z)$ of order 3, it follows easily that $3e_{n+3}=0$. But $e_{n+3} \neq 0$ for $\mathfrak{P}^1 g_n' \neq 0$ (by (6.2)). Therefore $2e_{n+3}$ has a representative map with range contained in S_n' since $\pi_{n+3}(S'_{n+2}) \approx Z_2$. Let $a_{n+3} \in \pi_{n+3}(S_n')$ be represented by this map. Then, it is clear that $a_{n+3} \neq 0$, any representative map of a_{n+3} is null homotopic in ϑ'_{n3}, and a_{n+3} generates $C(\pi_{n+3}(S_n'), 3)$ (since the injection homomorphism: $\pi_{n+3}(S_n') \to \pi_{n+3}(S_n' \vee S'_{n+2})$ is an into-isomorphism). It follows that every map: $\partial E_{n+4} \to S_n'$ representing an element of $C(\pi_{n+3}(S_n'), 3)$ extends in ϑ'_{n3} over E_{n+4}.

Now, since ϑ_{n3} and ϑ'_{n3} have the same homotopy type, we see easily that every map $f: \partial E_{n+1} \to S_n$ representing an element of $C(\pi_{n+3}(S_n), 3)$ extends in ϑ_{n3} over E_{n+4}. If F is such an extension, using (12.3) with $p=3$, the technique which identifies ∂E_{n+4} and S_n by f will lead to a proof of (13.10).

14. **On secondary obstructions.** The problem of the secondary obstruction of maps of a complex into a complex K was solved for the case that K is an n-sphere by Steenrod [10]. A generalization to the case that K is $(n-1)$-connected, $n \geqq 2$, was given by J. H. C. Whitehead [16]. We shall show here how the 2-fold symmetric product of K is related to this extension problem. (Our main purpose is to establish the proposition (14.6) for applications to the study of secondary obstructions of fiber bundles.)

Throughout this section, we denote by K an $(n-1)$-connected finite cellular complex, $n \geqq 2$, with sth skeleton K_s, X an arbitrary cellular complex with sth skeleton X_s and A a subcomplex of X. Let us recall from [15; 16] some properties of the Pontrjagin square and a square operation considered by J. H. C. Whitehead. Let ω: $S_{n+1} \to S_n$ be a fixed map between oriented spheres which represents the generator of $\pi_{n+1}(S_n)$, with Hopf invariant 1 in case $n=2$. Denote by $\Gamma_{n+1}(K)$ the subgroup of $\pi_{n+1}(K_{n+1})$ of elements represented by maps ωf: $S_{n+1} \to K_{n+1}$ where f is a map: $S_n \to K_{n+1}$. $\Gamma_{n+1}(K)$ is an invariant of K [15]. The operation: $H^n(X, A; \pi_n(K)) \to H^{n+2}(X, A; \Gamma_{n+1}(K))$ which is the Pontrjagin square $\mathfrak{p}_1$ for $n=2$ and is the square operation $\mathfrak{S}^2$ for $n \geqq 3$ was considered and studied in [15; 16]. For convenience, we shall write Sq_w^2 for this operation in both cases $n=2$ and $n \geqq 3$. Sq_w^2 possesses the following properties (14.1) and (14.2).

(14.1) *Let ϕ be a map of (X, A) into a pair (X', A') where X' is a cellular complex and A' a subcomplex of X'. Then $\phi^* Sq_w^2 = Sq_w^2 \phi^*$.*

(14.2) *Let h be a map of K' into K where K' is an $(n-1)$-connected finite cellular complex, and denote again by h the homomorphism: $\pi_s(K') \to \pi_s(K)$, $\Gamma_{n+1}(K') \to \Gamma_{n+1}(K)$ induced by h. Then, in (X, A), $\bar{h} Sq_w^2 = Sq_w^2 \bar{h}$ where h are the homomorphisms between cohomology groups (namely, $\bar{h}$: $H^n(X, A; \pi_n(K')) \to H^n(X, A; \pi_n(K))$, $H^{n+2}(X, A; \Gamma_{n+1}(K')) \to H^{n+2}(X, A; \Gamma_{n+1}(K))$) induced by h.*

The inclusion map: $K_{n+1} \to K$ gives naturally a homomorphism ν: $\Gamma_{n+1}(K) \to \pi_{n+1}(K)$. Let us write $Sq_w'^2 = \bar{\nu} Sq_w^2$: $H^n(X, A; \pi_n(K)) \to H^{n+2}(X, A; \Gamma_{n+1}(K))$ where $\bar{\nu}$ is the homomorphism: $H^{n+2}(X, A; \Gamma_{n+1}(K)) \to H^{n+2}(X, A; \pi_{n+1}(K))$ induced by ν. We see easily that (14.1) and (14.2) still hold if we replace Sq_w^2 by $Sq_w'^2$. In (14.2) if h is a homotopy equivalence, then h: $\Gamma_{n+1}(K') \approx \Gamma_{n+1}(K)$.

For simplicity, we shall write $\pi_s = \pi_s(K)$, $\Gamma_{n+1} = \Gamma_{n+1}(K)$, throughout the rest of this section and denote by κ_n the primary obstruction to contracting K, which is a cohomology class $\in H^n(K, \pi_n)$ (15).

(14.3) *In K, we have $Sq_w'^2(K_n) = 0$.*

For an open cell C in a cellular complex, we write for simplicity $\partial^\# C$ for the homotopy boundary(10) of C. By the properties of $Sq_w'^2$ recalled above we may prove (14.3) by considering any finite cellular complex K' which has the same homotopy of K, instead of K. Thus, by a statement given in the Appendix we may assume without loss of generality that K is composed of oriented open k-cells C_k's, etc., with

$$
\begin{aligned}
K_{n+2} = C_0 &\cup (\overset{1}{C}_n \cup \overset{2}{C}_n \cup \cdots \cup \overset{r}{C}_n) \\
&\cup (\overset{1}{C}_{n+1} \cup \overset{2}{C}_{n+1} \cup \cdots \cup \overset{s}{C}_{n+1} \cup \overset{s+1}{C}_{n+1} \cup \cdots \cup \overset{s'}{C}_{n+1}) \\
&\cup (\overset{1}{C}_{n+2} \cup \overset{2}{C}_{n+2} \cup \cdots \cup \overset{t}{C}_{n+2} \cup \overset{t+1}{C}_{n+2} \cup \cdots \cup \overset{t'}{C}_{n+2}),
\end{aligned} \tag{14.4}
$$

(15) See N. E. Steenrod, *Topology of fiber bundles*, Princeton University Press, 1951, p. 187.

where $(\alpha)\partial^{\#}C^{i}_{n+1}\subset C_0\cup C^{i}_{n}$ for $1\leqq i\leqq s$ (with $s\leqq r$) and is not null homotopic in $C_0\cup C^{i}_{n}$, (β) $\partial^{\#}C^{i}_{n+1}\subset C_0$ for $s+1\leqq i\leqq s'$, (γ) $\partial^{\#}C^{i}_{n+2}\subset K_n\cup C^{s+i}_{n+1}$ for $1\leqq i\leqq t$ (with $s+t\leqq s'$) and (δ) $\partial^{\#}C^{i}_{n+2}\subset K_n$ for $t+1\leqq i\leqq t'$. Since K_n is the union of r n-spheres contacting at one point, namely C_0, and $C_0\cup C^{j}_{n+1}$ for $s+1\leqq j\leqq s'$ is an $(n+1)$-sphere, $\partial^{\#}C^{i}_{n+2}$ for $1\leqq i\leqq t$ is homotopic in $K_n\cup C^{s+i}_{n+1}$ to a map f_i: $S_{n+1}\to K_n\cup C^{s+i}_{n+1}$ which maps a hemisphere E^{i+}_{n+i} in S_{n+1} into K_n and maps the complementary hemisphere E^{i-}_{n+1} into $C_0\cup C^{s+i}_{n+1}$ and $f_i(\partial^{\#}E^{i+}_{n+1})=f_i(\partial^{\#}E^{i-}_{n+1})=C_0$.

We shall consider the secondary boundary homomorphism α: $C_{n+2}(K, Z)\to\Gamma_{n+1}$ for K as given in [16, p. 71]. We assert that $\nu\alpha$: $C_{n+2}(K, Z)\to\pi_{n+1}$, regarded as a cochain with coefficient group π_{n+1}, is cohomologous to 0 in K. In fact, it is easy to see that $\nu\alpha$ has value 0 over $C^{t+1}_{n+2}, C^{t+2}_{n+2}, \cdots, C^{t'}_{n+2}$ and $\nu\alpha(C^{i}_{n+2})$ for $1\leqq i\leqq t$ is the element of π_{n+1} represented by the map f_i: $(E^{i-}_{n+1}, \partial E^{i-}_{n+1})\to(C_0\cup C^{s+i}_{n+1}, C_0)$. Regarding $C_0\cup C^{s+i}_{n+1}$ as an oriented sphere with orientation given by that of C^{s+i}_{n+1}, let n_i be the degree of this latter map. Then, $\nu\alpha(C^{i}_{n+2})$, $1\leqq i\leqq t$, is n_ia^i where a^i is the element of π_{n+1} represented by the identity map of $C_0\cup C^{s+i}_{n+1}$ into K. Let now $u_{n+1}=\sum_{1\leqq i\leqq t} a^iC^{s+i}_{n+1}$, which is an $(n+1)$-cochain of K with coefficient group π_{n+1}. It is readily seen that $\delta u_{n+1}=\nu\alpha$. We use then the relation $\{\alpha\}=Sq^2_w(\kappa_n)$ in [16, p. 79] to complete the proof of (14.3).

We shall write E_{n+2} for an oriented $(n+2)$-cell with boundary = the oriented sphere S_{n+1}. We regularly imbed S_n into $S_n * S_n$ and K into $K*K$ in the sense of §3, and write

$$S_n\subset S_n*S_n,\qquad K\subset K*K.$$

(14.5) *Let $K=S_n(n\geqq 2)$. Then, by* (13.3), *ω: $S_{n+1}\to S_n$ extends in S_n*S_n over E_{n+2}. If h is such an extension, we have that $h^*\xi^{-1}Sq'^2_w\mu(\kappa_n)\in H^{n+2}(E_{n+2}, S_{n+1}; \pi_{n+1}(S_n))$ is represented by the cochain $[\omega]E_{n+2}$ where $[\omega]$ is the element of $\pi_{n+1}(S_n)$ represented by ω, ξ is the isomorphism* (13.1), *and μ is the into-isomorphism*: $H^n(S_n, \pi_n(S_n))\to H^n(S_n*S_n, \pi_n(S_n))$.

In the case $K=S_n$, $Sq'^2_w=Sq^2_w$. We may deduce easily (14.5) from (13.4)([16]).

Let us write the cohomology sequence of the pair $(K*K, K)$ with coefficient group G as

$$\cdots\to H^s(K*K, K; G)\xrightarrow{\xi} H^s(K*K, G)\xrightarrow{\eta} H^s(K, G)$$
$$\xrightarrow{\delta} H^{s+1}(K*K, K; G)\to\cdots.$$

It follows from the exactness of this cohomology sequence and (14.3) that there exists a cohomology class $u_{n+2}\in H^{n+2}(K*K, K; \pi_{n+1})$ such that $\xi(u_{n+2})$

([16]) Here, the identity $2p_1u=u\cup u$ in [16, p. 76] is useful for our purpose. Note that the cup product $u\cup u$ in this identity is defined in terms of the pairing $g\cdot g'$ = the J. H. C. Whitehead product $[g, g']$. See also the footnote in [16, p. 76].

$=Sq_w'^2\mu(K_n)$ where μ is the into-isomorphism: $H^n(K, \pi_n) \rightarrow H^n(K * K, \pi_n)$ given in (3.2).

(14.6) *Let* $u_{n+2} \in H^{n+2}(K * K, K; \pi_{n+2})$ *be such that* $\xi(u_{n+2}) = Sq_w'^2\mu(\kappa_n)$. *If* $f: S_{n+1} \rightarrow K$ *is a map which represents an element of* Γ_{n+1}, *then* f *extends in* $K * K$ *over* E_{n+2}. *Moreover, if* F *is such an extension, we have that* $F^*(u_{n+2}) \in H^{n+2}(E_{n+2}, S_{n+1}; \pi_{n+1})$ *is represented by the cochain* $[f]\ E_{n+2}$ *where* $[f]$ *denotes the element of* π_{n+1} *represented by* f.

In fact, by replacing a map: $S_{n+1} \rightarrow K$ homotopic to f if necessary, we may assume that $f = f'\omega$ where f' is a map: $S_n \rightarrow K$. We may assume also that f' maps the reference point in S_n used in the regular imbedding of S_n in $S_n * S_n$ to the reference point in K used in the regular imbedding of K in $K * K$. It follows that $g = f' * f'$ is a map: $(S_n * S_n, S_n) \rightarrow (K * K, K)$ such that $g | S_n = f'$. Now, ω has an extension $h: E_{n+2} \rightarrow S_n * S_n$, so $F_0 = gh: E_{n+2} \rightarrow K * K$ is a desired extension of f. Since $F_0^*(u_{n+2}) = h^*g^*(u_{n+2})$ with $g^*: H^{n+2}(K * K, K; \pi_{n+1}) \rightarrow H^{n+2}(S_n * S_n, S_n; \pi_{n+1})$, $h^*: H^{n+2}(S_n * S_n, S_n; \pi_{n+1}) \rightarrow H^{n+2}(E_{n+2}, S_{n+1}; \pi_{n+1})$, using (14.5) and (14.2) for $Sq_w'^2$ we verify that $F_0^*(u_{n+2})$ is represented by $[f]\ E_{n+2}$. Finally, let F be an arbitrary extension of f in $K * K$ over E_{n+2}. We assert that $F^*(u_{n+2}) = F_0^*(u_{n+2})$. In fact, let E_ω be the cellular complex with nth skeleton S_n, constructed by adjoining E_{n+2} to S_n by means of the map ω. Then, F and F_0 give rise naturally to maps F', F_0' : $E_\omega \rightarrow K$ such that $F' | S_n = f = F_0' | S_n$, and hence $F'^*Sq_w'^2\mu(K_n) = F_0'^*Sq_w'^2\mu(\kappa_n)$ $(= Sq'^2 f^*(\kappa_n)$, by (3.6) and (14.1) for $Sq_w'^2)$. Our assertion follows then from the fact that $H^{n+2}(E_\omega, \pi_{n+1})$ and $H^{n+2}(E_{n+2}, S_n; \pi_{n+1})$ are isomorphic in a natural manner. This completes the proof of (14.6).

Finally, let us give the following example which shows how to use (14.6) in the extension problem straightforwardly.

(14.7) *Let* f *be a map*: $X_{n+1} \rightarrow K$ *with* $X =$ *an* $(n+2)$*-dimensional simplicial complex. Then, the secondary obstruction*

$$Z^{n+2}(f) \in H^{n+2}(X, \pi_{n+1})$$

is given by

$$Z^{n+2}(f) = Sq_w'^2 f^*(\kappa_n).$$

As in the proof of (14.3), to see (14.7) we may consider K as a cellular complex composed of oriented open cells with K_{n+2} written as in (14.4). We may assume also that f is a cellular map. Define $f': X_{n+1} \rightarrow K$ by taking $f'(x) = C_0$ if $x \in X_0 = f^{-1}(C_{n+1}^{s+1} \cup C_{n+2}^{s+2} \cup \cdots \cup C_{n+2}^{s'})$, and taking $f'(x) = f(x)$ if $x \notin x_0$. Since $f^{-1}(C_0)$ separates X_{n+1}, f' is a map. Also, $f' | X_n = f | X_n$, so $f'^*(\kappa_n) = f^*(\kappa_n)$, $Z^{n+2}(f') = Z^{n+2}(f)$, and $f' | \partial\sigma_{n+2}$ represents an element of Γ_{n+1} for any oriented $(n+2)$-simplex σ_{n+2} in X[17].

---

[17] To see this we verify first that the boundary homomorphism: $\pi_{n+1}(K_n \cup C_{n+1}^1 \cup \cdots \cup C_{n+1}^s, K_n) \rightarrow \pi_n(K_n)$ is an into-isomorphism.

By (14.6), f' is extendable to a map $F: X \to K * K$, and the commutativity in each square of the diagram

$$\begin{array}{ccccc} H^s(K*K, K; G) & \xrightarrow{\xi} & H^s(K*K, G) & \xrightarrow{\eta} & H^s(K, G) \\ \downarrow F^* & & \downarrow F^* & & \downarrow F'^* \\ H^s(X, X_{n+1}; G) & \xrightarrow{\xi} & H^s(X, G) & \xrightarrow{\eta} & H^s(X_{n+1}; G) \end{array}$$

holds, where the upper homomorphisms ξ and η were given before, and the lower ξ and η are respectively the injection and projection homomorphisms in the cohomology sequence of (X, X_{n+1}). Thus, since, by (14.6), for any oriented $(n+2)$-simplex σ_{n+2}, $(F|\sigma_{n+2})^*(u_{n+2}) \in H^{n+2}(\sigma_{n+2}, \partial\sigma_{n+2}; \pi_{n+1})$ is represented by $[f'|\partial\sigma_{n+2}]\sigma_{n+2}$, $F^*Sq'^2_w\mu(K_n) = F^*\xi(u_{n+2}) = \xi F^*(u_{n+2}) =$ the element of $H^{n+2}(X, \pi_{n+1})$ represented by $\sum_{\sigma_{n+2}\subset X}[f'|\partial\sigma_{n+2}]\sigma_{n+2}$, namely, the secondary obstruction $Z^{n+2}(f')$. But, $F^*Sq'^2_w\mu(\kappa_n) = Sq'^2_w F^*\mu(\kappa_n) = Sq'^2_w f'^*(\kappa_n)$ (by (3.6)). This completes the proof of (14.7).

We may deduce in a similar manner as above and by arguments in [16, p. 78] the relative extension theorem on secondary obstructions.

Appendix

J. H. C. Whitehead, in Comment. Math. Helv. vol. 22 (1949) pp. 48–92 and Annales de la Société Polonaise de Mathématiques vol. 21 (1948) pp. 176–186, considered a particular type of A^2_n-complex, called a reduced A^2_n-complex, and showed that every A^2_n-complex is of the same homotopy type of a reduced one. This notion can be generalized. We give the exact formulation of this generalization in the following.

Consider a cellular complex K composed of open n-cells e^i_n's with characteristic maps $f_{ni}: \sigma_n \to \bar{e}^i_n$ as given in [14, p. 221] satisfying the following condition (Δ), namely,

(Δ) *For each e^i_n, in the boundary $\partial\sigma_n$, there are a finite number of non-overlapping cells*(8), *say $\tau^1_{n-1}, \tau^2_{n-1}, \cdots, \tau^r_{n-1}$, such that* ($\alpha$) *for each j, we have a homeomorphism $g_j: \sigma_{n-1} \to \tau^j_{n-1}$ and $(f_{ni}|\tau^j_{n-1})g_j$ is the characteristic map of some e^l_{n-1}, and that* (β) *$f_{ni}(\partial\sigma_n - \bigcup_{j=1}^r \tau^j_{n-1}) \subset K_{n-2}$ (= the $(n-2)$th skeleton of K).*

For a given pair $(e^i_n, e^{i'}_{n-1})$, we denote by $[e^i_n, e^{i'}_{n-1}]$ the total number of τ^j_{n-1}'s, say $\tau^{j_1}_{n-1}, \tau^{j_2}_{n-1}, \cdots, \tau^{j_s}_{n-1}$, such that $(f_{ni}|\tau^j_{n-1})g_j$ is the characteristic map of $e^{i'}_{n-1}$. ($[e^i_n, e^{i'}_{n-1}]$ may be zero.) Suppose that fixed orientations are assigned to the open cells of K. Let σ_n, $\tau^{j_k}_{n-1}$ be oriented coherently with f_{ni}, $(f_{ni}|\tau^{j_k}_{n-1})g_{j_k}$ respectively. Then $\tau^{j_k}_{n-1}$ appears as a face in σ_n positively or negatively, and we write $\{e^i_n, \tau^{j_k}_{n-1}\} = 1$ or -1 according as the former or the latter is the case. We write $\{e^i_n, e^{i'}_{n-1}\} = \sum_{k=1}^s \{e^i_n, \tau^{j_k}_{n-1}\}$. It may be called the incidence number of the pair $(e^i_n, e^{i'}_{n-1})$ and the homology boundary of e^i_n will be given by $\partial e^i_n = \sum_{i'} \{e^i_n, e^{i'}_{n-1}\} e^{i'}_{n-1}$.

We shall call a simply-connected finite cellular complex K composed of oriented n-cells e_n's a *reduced complex* if it satisfies the condition (Δ) and satisfies also the following further conditions (i) and (ii):

(i) $[e_n, e_{n-1}] = \pm \{e_n, e_{n-1}\}$ *for any pair* (e_n, e_{n-1}).

(ii) *The* e_n*'s can be numbered as*

$$\overset{1}{e_n}, \overset{2}{e_n}, \cdots, \overset{i_n}{e_n}, \overset{i_n+1}{e_n}, \cdots, \overset{i_n'}{e_n}, \overset{i_n'+1}{e_n}, \cdots, \overset{i_n''}{e_n}, \qquad (n = 0, 1, 2, \cdots),$$

with $i_0'' = 1$, $i_1' = 0$ *and* $i_n'' - i_n' = i_{n-1}' - i_{n-1} = j_n$ *such that the homology boundary* $\partial e_n^i = 0$ *for* $1 \leqq i \leqq i_n'$ *and* $\partial e_n^{i_n'+i} = \theta_{n-1,i} e_{n-1}^{i_{n-1}+i}$ *with* $|\theta_{n-1,i}| \geqq 2$ *for* $1 \leqq i \leqq j_n$.

Under these conditions, we see easily that the nth Betti number of K is i_n, the nth torsion coefficients (with respect to homology) are $\theta_{n1}, \theta_{n2}, \cdots, \theta_{nj_{n+1}}$, and the homotopy boundary([10]) of e_n^i for $1 \leqq i \leqq i_n'$ lies in K_{n-2}.

We have:

Every connected, simply-connected finite cellular complex is of the same homotopy type of some reduced complex.

This statement can be proved by some standard procedure in combinatorial homotopy theory. We wish to omit the detail here. The guiding idea in proving this may be the same as the reduction of incidence matrices of a finite complex into canonical form by elementary transformations([18]). We may begin by taking the given cellular complex as simplicial [14, p. 239], so that it satisfies the condition (Δ). Besides, arguments such as follows will be useful. Let K be an n-dimensional cellular complex which satisfies the condition (Δ) and whose secondary skeleton is the union of 2-spheres contacting at one point. Then, (α) with $e_n^i, e_{n-1}^{i'}, \tau_{n-1}^j$'s given as before, if $\{e_n^i, \tau_{n-1}^{j'}\} = 1$, $\{e_n^i, \tau_{n-1}^{j''}\} = -1$, we may deform the homotopy boundary of e_n^i without altering the homotopy type of K such that the number $[e_n^i, e_{n-1}^{i'}]$ is diminished by 2 and the resulting cellular complex still satisfies the condition (Δ). Next, (β) if $[e_n^i, e_{n-1}^{i'}] = 1$ and $[e_n^{i''}, e_{n-1}^{i'}] = 0$ for all $e_n^{i''} \neq e_n^i$, the subcomplex $K - e_n^i - e_{n-1}^{i'}$ is a deformation retract of K.

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